USER COMMENTS
M&M - DEATH AND IMMIGRATION

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TEACHER COMMENTS

1 Preparation

I was assigned to teach our Ordinary Differential Equations course in Spring 2015. I had not taught the course since Spring 2008. I used the classic Boyce and DiPrima [1] textbook then. This time I decided to try an "open textbook," *Notes on Diffy Qs: Differential Equations for Engineers* by Jiří Lebl [2]. Lebl's book is available as a PDF download; one can also buy a printed copy from `lulu.com` (for around $16). I wanted to try a less-expensive textbook, and this one covered all of the topics that our course requires.

Before the start of the Spring 2015 semester, I looked over the different activities that are available on the SIMIODE Web site. Of the numerous activities, I decided that I might try three activities:

1. M&M - Death and Immigration `<https://www.simiode.org/resources/121>`
2. Water Flowing from Tank `<https://www.simiode.org/resources/288>`
3. Spring Mass Damping Data Analysis `<https://www.simiode.org/resources/804>`

I had never used activities to introduce topics before, and I had no intention of trying to build an entire course around activities. I decided that these three would allow me to gain some experience in this different mode of teaching differential equations. I decided that M&M would be a first-day activity, before we opened the textbook, and that the other two would be used at the appropriate points in the course. Still, I was nervous, and so I also decided that if M&M did not go well, then I would not do the other two activities.

That was the extent of my planning. There was no apparent integration between any of the activities and the textbook. I would be shooting from the hip once the semester started.

2 Classroom Experience

We did the M&M activity on the first day of class, right after reviewing the syllabus and before starting any discussion of differential equations (DE).

For the first part, death without immigration, we went through (a)–(e) just fine. When it came to the mathematical model, most of the groups (they worked in pairs) came up with $a(n) = 0.5a(n-1)$, $a(0) = 50$; two groups came up with $a(n) = 0.5^n(50)$. A class discussion of whether these two models,

$$a(n) = 0.5a(n-1) \quad \text{and} \quad a(n) = 0.5^n(50),$$

are equivalent or, if not, which one was correct ensued. We all agreed that they are equivalent, and that both modeled the data.

I found that the students naturally came up with models $a(n) = \ldots$ instead of $a(n+1) = \ldots$, but they easily accepted that we could change variables from n to $n+1$ in their model.

For the second part, death with immigration, we went through (i)–(m) just fine. When it came to the model, they naturally recorded the number in the cup before adding 10 M&Ms; they called the number in the cup $a(n)$. When they tried to come up with a mathematical model, however, they decided that it was easier to record the number in the plate, that is, $a(n) + 10$, so they let $b(n) = a(n) + 10$. The two groups that earlier came up with $a(n) = 0.5^n(50)$ for death without immigration, first wrote $b(n) = 0.5^n(50) + 10$. The other groups wrote $b(n) = a(n) + 10$, and I had to suggest that they introduce $b(n-1)$ instead of $a(n)$ in their model. After a little discussion, these other groups arrived at $b(n) = 0.5b(n-1) + 10$. A class discussion of whether the two models,

$$b(n) = 0.5^n(50) + 10 \quad \text{and} \quad b(n) = 0.5b(n-1) + 10,$$

are equivalent or, if not, which one was correct then ensued. After plugging in some values of n, it was decided that $b(n) = 0.5^n(50) + 10$ was not a good model because the data was settling down around 20 M&Ms, whereas the model tended to 10. Eventually, we all agreed that $b(n) = 0.5b(n-1) + 10$ was a good model. I gave them as an exercise to come up with a closed form formula for $b(n)$ just like the two groups had come up with $a(n) = 0.5^n(50)$ for death without immigration. I went through a few steps with them,

$$\begin{aligned}
b(1) &= 0.5b(0) + 10,\\
b(2) &= 0.5b(1) + 10 = 0.5[0.5b(0) + 10] + 10,\\
b(3) &= 0.5b(2) + 10 = 0.5[0.5[0.5b(0) + 10] + 10] + 10,
\end{aligned}$$

and asked them to try to continue and to see a pattern.

I think that the students were surprised that the data tended to 20 and that the model $b(n) = 0.5b(n-1)+10$ seemed to agree with that when the model has a "$+10$" instead of "$+20$" in it. We then discussed if they thought both $b(n)$ and $b(n-1)$ tended to the same value if were a limiting value. They agreed that they would, and so we looked at solving

$$b(n) = 0.5b(n-1) + 10 \quad \text{as} \quad n \to \infty, \quad \text{where} \quad b(n),\ b(n-1) \to E,$$

and marvelously they saw that $E = 20$. That was very pleasing.

For both parts, death without immigration and death with immigration, I think that students naturally came up with a model $b(n) = \ldots$. Following the death with immigration part, no one thought to look at the difference $b(n) - b(n-1)$ since the model already seemed to agree with the data. I had to ask them to consider the difference $b(n) - b(n-1)$, but I did not motivate why we would want to look at the difference. After introducing the difference $b(n) - b(n-1)$, I suggested that we change variables from n to $n+1$ so that it would match the handout. We then basically followed the handout in going from a discrete model to a continuous model. I later pointed out that if $\Delta n = 1$, then the model $b(n+1) - b(n) = -0.5b(n) + 10$ can be written

$$b(n + \Delta n) - b(n) = -0.5b(n)\Delta n + 10\Delta n,$$

which is consistent with

$$b(t + \Delta t) - b(t) = -0.5b(t)\Delta t + 10\Delta t$$

that we had come up with. Going from there to the DE $b' = -0.5b + 10$ was not too hard after we recalled the limit definition of the derivative.

We tried to solve the DE. Everyone said to integrate. They got

$$b = -0.5b^2/2 + 10b + C.$$

Using $b(0) = 50$, they then found C, and then they used the quadratic formula to solve for $b(t)$. I let them go through all that, and at the end I asked them if the solution $b(t) =$ constant made sense. They concluded that it did not because $b(t) =$ the number of M&Ms was not constant for all time. So we traced backward through their solution steps to find the mistake, and they eventually realized that $\int(-0.5b(t) + 10)dt \neq -0.5b^2/2 + 10b$ because the integral is w.r.t. t and not w.r.t. b. I think that was a very useful reminder for them. At this point I told them that the DE is an example of a first-order linear DE (in $b(t)$), and that we would learn how to solve such equation in a couple of sections.

I did not complete the activity with the part on "Moving to a new reality" (a microorganism in a Petri dish). We had already used 1.25 class periods (collecting data and discussing the models, up to attempting to solve the differential equation), and I had to move on.

3 Introducing first-order linear differential equations

Fast forward a couple of sections in the textbook, and to introduce solving first-order linear differential equations I had them first recall their model for death with immigration,

$$b' = -0.5b + 10, \quad b(0) = 50.$$

We used that equation to develop a method for solving $y'(t) + p(t) \cdot y(t) = q(t)$ in general using an integrating factor. Here, we found the solution,

$$b(t) = 20 + 30e^{-0.5t},$$

and it was really nice to see that $b(t) \to 20$ as $t \to \infty$ just as the data had born out. All in all, it was quite satisfying to me to be able to refer back to their model from the activity.

4 Student comments

A few students provided written feedback on the M&M activity.

- I enjoyed the M&M activity, I'm a very hands on learner. It was nice to do something different from the usual book work and notes. I would encourage more visual exercises and handouts like the M&M activity offered. The activity was also stimulating and engaging.
- I felt the M&M activity was engaging. I liked working in pairs because I felt more engaged compared to doing the entire activity as a class.
- I thought the M&M activity was very helpful in getting the students adjusted to the thought process we needed to have while taking diffy q. It took a bit for me to see the point of the activity, but it really helped in recognizing how differential equations can be applied. I looked back through the packet and couldn't find something you had brought up that actually helped, although it wasn't a big theme throughout the course. It was the situation where a term in a function gradually goes to zero as time goes on and we are left with another term that is unchanged. The steady state maybe? Anyways, like I said, it was helpful seeing that even if it only showed up once or twice after that first day.
- I at first I didn't really understand why we were doing the activity, but when you had us try to develop a formula for what was occurring, it became clearer and clearer. It was fun activity that was interactive and i liked how you set it up to be able go back to it as we were learning different lessons. That, I would say, was the most helpful part about it. Having us see how simple what should seem like random activity can be developed into a differential equation.

5 A colleague's comments

I was fortunate to have a colleague visiting my class during the M&M activity. The students were to work in pairs, but there was an odd number of students, so he paired with one of the students. Here are his brief remarks about the activity, thinking back upon it a few months later.

I thought the activity was a good way of introducing differential equations. The level seemed appropriate. The questions were very well thought out (e.g. stating and evaluating the importance of the assumptions we need to make about the experiment). It would have been nice to have room in the handout itself to answer the questions (though it would have made for a longer handout).

I would use this activity given its quality and would also consider using other activities from SIMODE (upon inspection of course).

6 Postmortem

There are a few things that I would do differently the next time I use this activity.

1. After the students came up with two models for death without immigration,
$$a(n) = 0.5a(n-1) \quad \text{and} \quad a(n) = 0.5^n(50),$$
we reasoned that both models are equivalent, and then moved on. I think that next time I should pause to look at $a(n) = 0.5a(n-1)$ more carefully, and to obtain a separable DE. Specifically, it might be interesting at this point to lead the students through
$$\begin{aligned} a(n) = 0.5a(n-1) \quad &\rightarrow \quad a(n+1) = 0.5a(n) \quad \rightarrow \quad a(n+1) - a(n) = -0.5a(n) \\ &\rightarrow \quad a(t+1) - a(t) = -0.5a(t)\Delta t \quad \rightarrow \quad \frac{da}{dt} = -0.5a. \end{aligned}$$
This might be a nice discussion to prepare the students for developing the 1st-order DE in the death with immigration part.
2. Recall that I gave them as an exercise to come up with a closed form formula for $b(n) = 0.5b(n-1) + 10$. Unfortunately, I never returned to review this with them, and so I do not know if they ever completed this exercise. I think that this is a useful exercise, and, so, in the future I will review this with them. I think it would be useful to see how the first few $b(n)$'s can lead us to conjecture that
$$b(n) = 0.5^n b(0) + 10(0.5^{n-1} + 0.5^{n-2} + \cdots + 0.5 + 1).$$
This would provide an opportunity to review the geometric sum, leading us to
$$b(n) = 0.5^n(30) + 20 = 2^{-n}(30) + 20 = 2^{1-n}(15) + 20,$$
where we have used $b(0) = 50$. This, in turn, would provide an opportunity to review proof by induction.
3. Recall that in developing the continuous model from $b(n) = 0.5b(n-1) + 10$, I had asked them to consider the difference $b(n) - b(n-1)$, but I did not motivate why we would want to look at the difference. I think that next time I need to motivate why we would want to look at the difference when we already had a seemingly perfectly good model.

The M&M activity went well, in my estimation, and I had every intention of using the other two activities that I had chosen. Unfortunately, time got away from me, and abandoned the other two activities. The next time I have an opportunity to teach DE again, I will try to plan better so that I will be able to use all three activities. After that, who knows, I may try to build an entire course around activities.

REFERENCES

[1] Boyce, William E. and Richard C. DiPrima. 2005. *Elementary Differential Equations and Boundary Value Problems, with ODE Architect CD*, 8th ed., New York: Wiley.

[2] Lebl, Jiří. 2015. *Notes on Diffy Qs: Differential Equations for Engineers* `<http://www.jirka.org/diffyqs/>`.