STUDENT VERSION
ONE MASS SPRING

Eric Ahlstrom, Mathematics
Itasca Community College
Iron Range Engineering
Grand Rapids MN USA

Swarn Singh, Mathematics
Sri Venkateswara College
University of Delhi
New Delhi INDIA

John T. Sieben, Mathematics
Texas Lutheran University
Seguin TX USA

Tiernan Fogarty, Mathematics
Oregon Institute of Technology
Klamath Falls OR USA

J. B. Thoo, Mathematics
Yuba College
Marysville CA USA

STATEMENT

We are interested in the motion of a mass (bob)-spring system that is hanging vertically as shown in Figure 1. The goal is to come up with a mathematical model of the motion (vertical position) of the bob as a function of time.

In Figure 1, we see a bob hanging from a spring vertically, and below it is a motion detector that was used to collect data (vertical position of the bob and time) for the motion of the bob. This data was then compared against the mathematical model for the motion.

After either raising the bob or pulling it down from its rest position or static equilibrium, and then letting go, it moves up and down like clockwork. Let us assume that there are only two forces acting on the bob: the force that is due to gravity and the force produced by the spring. We will not assume there is resistance force. The force that is due to gravity is given by mass $\times$ gravity

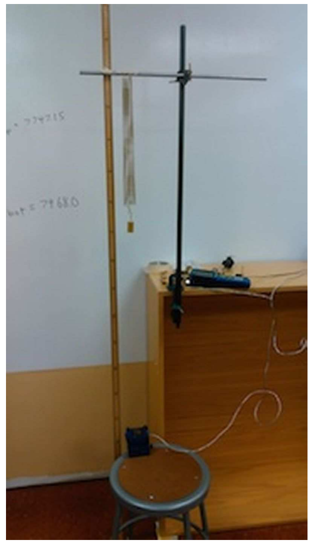

Figure 1. A set up of a single mass (bob)-spring system. You can see the motion detector sitting on the stool in the photograph. Here, the bob has a mass of 200 grams, but the spring constant is unknown.

(mg), and the force produced by the spring is proportional to the amount (length) that the spring has been stretched or compressed (kx) from its rest position (this is called *Hooke's law* [1]). The constant k in the spring force is called the *spring constant*, and it has units dyne/cm.

1 Mathematical Model

A bob is attached to a spring that is hanging vertically. See Figure 1.

1.1 Activity 1

Drawn in Figure 2 is a free-body diagram to analyze the forces that are acting on the bob. Take the positive direction to be down for convenience.

The hanging spring without a bob attached has length L cm, and the spring with a bob attached (but not in motion) has length $(L + a)$ cm. Fill in the blanks below.

Do we need to make any assumptions about the dimensions of the bob? Explain.

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At rest or static equilibrium, the spring force equals weight force and this implies $kx =$ ____________

Now the bob is moving (straight up and down, with no side to side swaying), and at time t s it is at position $x(t)$ cm or just x cm. Let the spring constant be k and the acceleration due to gravity be g.

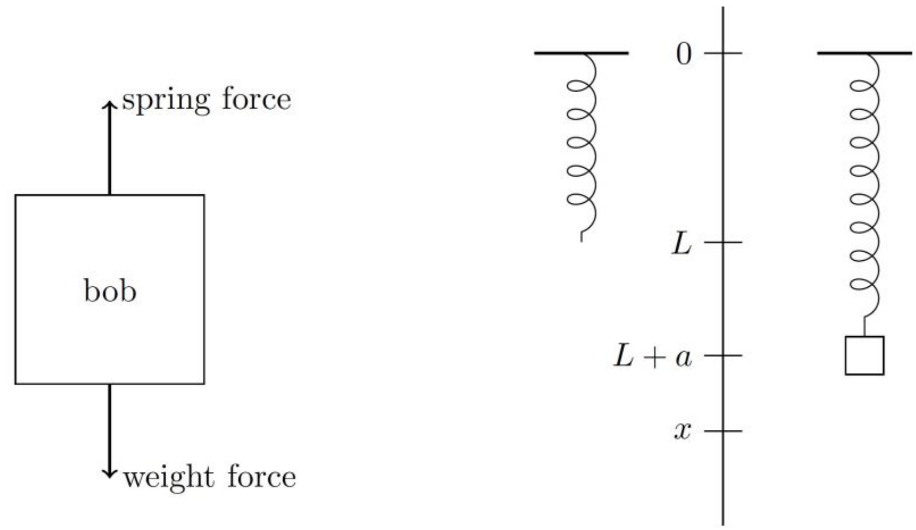

Figure 2. A free-body diagram to analyze the forces that are acting on bob that is hanging on the end of a spring.

What are the units for the acceleration due to gravity? ____________________
What are the units for the spring constant? ____________________
How might you calculate k if it is not known to you? ____

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According to Newton's second law of motion, the equation of motion of the bob is

$$\text{mass} \times \text{acceleration of the bob} = \text{total forces acting on the bob}$$
$$mx'' = \underline{\hspace{8cm}}$$

What happened to the term with g, the acceleration due to gravity? Does this make sense?

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1.2 Activity 2

The equation of motion for the bob is then

$$mx'' = -kx. \tag{1}$$

Can you justify this differential equation in your own words?

To simplify what is to come, divide through the equation by m and bring all the terms to the left-hand side of (1):

$$x'' + \frac{k}{m}x = 0 \quad \text{or} \quad x'' + \omega^2 x = 0, \tag{2}$$

where $\omega = \sqrt{k/m}$ (taking the positive square root).

Remember that $x = x(t)$ is a function of t. What might be a solution of the equation? Think about the motion of the bob: how might you describe the motion in one word?

The bob moves up and down like clockwork; the motion of the bob is ________________.

If you suggested that the motion of the bob is periodic, what periodic function comes to mind immediately?

When someone says, "periodic function," I think ________________ or ________________ immediately.

So, let us assume that the solution of the equation of motion is

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \tag{3}$$

where c_1 and c_2 are arbitrary constants. Verify that (3) is indeed a solution of (2), $x'' + \omega^2 x = 0$. Show that the frequency terms must be ωt for this motion.

To see why the solution (3) is a combination of a cosine function and sine function, describe the position of the bob at time $t = 0$ in the following cases.

$c_1 = 0, c_2 \neq 0$: ____

$c_1 > 0, c_2 = 0$: ____

$c_1 < 0, c_2 = 0$: ____

Remark: One may use a computer algebra system, such as Mathematica or Maple or Sage, to solve the equation of motion. We will also learn how to solve such "second-order" equations by hand very soon, but not in this activity.

You can see that to solve the equation $x'' + \omega^2 x = 0$ one might require two integrations because of the second derivative, which would result in two constants of integration. This is why we have two unknowns, c_1 and c_2, in the solution. The constants would be determined by some known "initial conditions" (that might be measured in the experiment). The initial conditions are some known information about the mass (bob)-spring system at a particular time (usually at time 0 seconds, whence *initial* conditions).

What are some reasonable initial conditions that would help us determine c_1 and c_2?

- room temperature
- time of the day

- date
- position of the bob
- velocity of the bob
- humidity in the room

We take as initial conditions the position and the velocity. Find c_1 and c_2 if we suppose that

$$x(0) = x_0, \quad x'(0) = x_1.$$

The values will be in terms of x_0 and x_1.

$$c_1 = \underline{\hspace{3cm}}, \quad c_2 = \underline{\hspace{3cm}}$$

Now, using trigonometric identities, we may rewrite the solution (3) of (2) as

$$x(t) = A\cos\bigl(\omega(t-\tau)\bigr).$$

Use trigonometric identities to obtain $A\cos\bigl(\omega(t-\tau)\bigr)$ from $c_1\cos(\omega t) + c_2\sin(\omega t)$, and verify that this is indeed a solution of $x'' + \omega^2 x = 0$. Here, A is the amplitude, ω is the frequency, and τ is the shift in time when the "clean data" starts (at a peak). (By "clean data" we mean that the first several entries of the collected data that appear to be spurious have been deleted if necessary due to unsure release of the bob with the hand.) Thus, k is unknown for this activity. We shall try to approximate k from the data.

1.3 Activity 3

Set up an experiment similar to the one that is shown in Figure 1 to collect some data for the position of the bob of mass m at time t for, say, 20 or 30 seconds. (That should be more than enough data.) Record the data in a spreadsheet, for example, and clean it up.

If you are unable to set up an experiment, you may use the data that we collected; it is available in the accompanying file `3-90-rfcut2.csv`. Note that this data has been cleaned up. In our experiment, the bob has mass 200 g. This data is plotted in Figure 3.

Unfortunately, the student who conducted the experiment (not you, of course) forgot to note the spring constant k before putting away all of the equipment, tossing the spring into a box that contains several similar-looking springs. Describe (in words) how you might approximate k using the data collected and the mathematical model:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}(t-\tau)\right).$$

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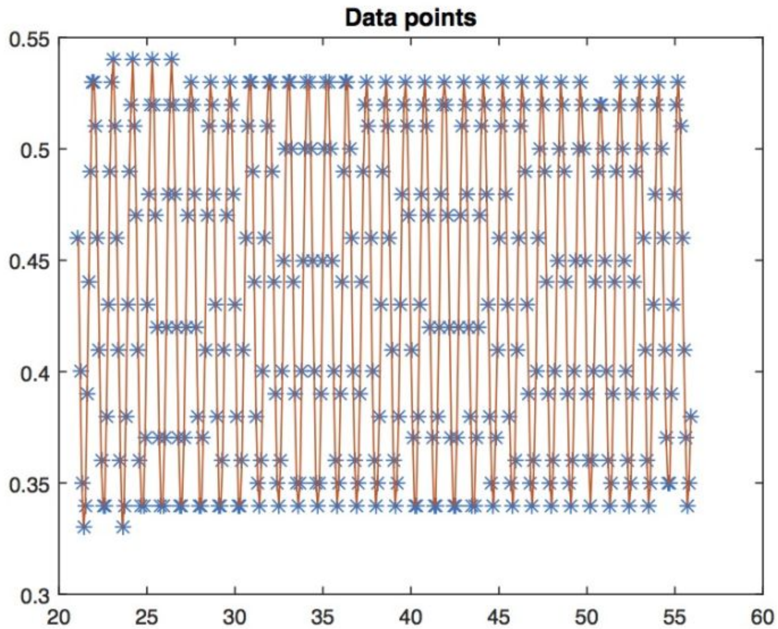

Figure 3. The plot of the data that is contained in the file 3-90-rfcut2.csv. The data points are marked as stars. By tracing the red graph that connects consecutive data points, we can see the periodic motion of the hanging bob.

Now do it: Approximate k, and then plot the solution from the mathematical model and compare it to a plot of the collected data. Did your model do a good job of reproducing the data? Define "good" here. You will have to estimate τ. To do this, you may plot your data, note when it becomes approximately periodic, and let τ be a time at a peak of this periodic part.

REFERENCES

[1] Wikipedia. Hooke's Law <https://en.wikipedia.org/wiki/Hooke%27s_law>. Accessed July 17, 2015.