TEACHER VERSION

SIMPLE PENDULUM WITH INITIALCONDITIONS

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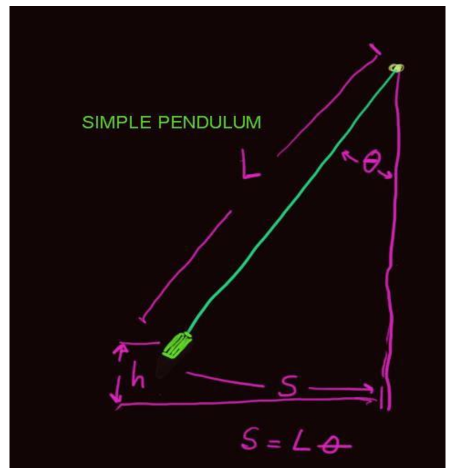
STATEMENT

The student is asked to derive and solve a differential equation that gives the position (angle) of a pendulum bob as a function of time t. One relationship that allows us to express the angle θ as a function of time t comes from the conservation of energy principle of physics. The pendulum's motion is a constant tradeoff between kinetic energy and potential energy. Indeed, the sum of the potential energy and the kinetic energy of the pendulum remain constant for an ideal (no or almost no friction effect) pendulum.



Potential energy (PE) = mgh where m is the mass of the pendulum bob, g is the acceleration due to gravity, and h is the height of the bob from its lowest point. Kinetic Energy (KE) = $\frac{1}{2}mv^2$. Conservation of energy suggests that the sum of these two terms will be constant.

$$mgh + \frac{1}{2}mv^2 = C$$

$$mgh + \frac{1}{2}m(\frac{ds}{dt})^2 = C,$$

and note that s=L θ and $\mathrm{h} = \mathrm{L}(1\text{-}\cos(\theta\ ))$which gives

$$mgL(1 - \cos(\theta) + \frac{1}{2}m(L\frac{d\theta}{dt})^2 = C.$$

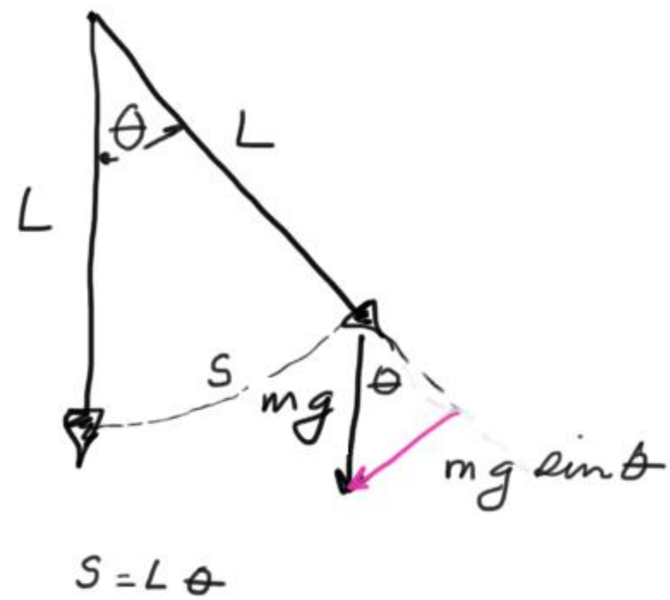
Differentiate with respect to t (don't forget the chair rule) and obtain

$$mL^2\frac{d\theta}{dt}\left[\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta)\right] = 0\ ,$$

from which we infer

$$\left[\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta)\right] = 0 \qquad (1)$$

As an alternate approach one could start with Newton's Second Law of Motion, F=m*a, which asserts that the total force acting on a body is its mass times its acceleration. We apply this along the tangential direction to the arc traveled by the mass (pendulum bob).



From F=ma it follows that

$$m * L\frac{d\theta^2}{dt^2} = -mg * \sin(\theta)$$

(the negative sign because this is a restoring force.) Upon algebraic manipulation this becomes

$$\left[\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta)\right] = 0$$

which agrees with (1)

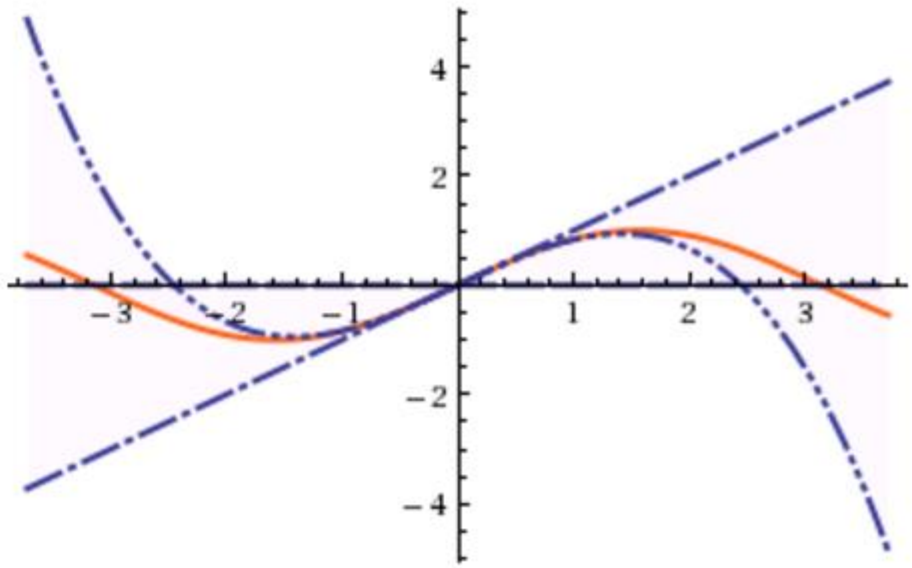
This is a non-linear second order differential equation but can be simplified by substituting θ for $\sin(\theta)$.The graph below shows two y=$\sin(\theta)$ and two Taylor polynomials y= θ and $y = \theta - \frac{\theta^3}{6}$ plotted on the same axes. This suggests that for small values of θ, $\sin(\theta)$ can reasonably be replaced by θ. This substitution will greatly simplify solution of (1).

$$\sin(\theta) \approx \theta \text{ for small angles } \theta.$$

$\left[\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta\right] = 0$ is recognized as a linear second order equations with constant coefficients. By finding the characteristic polynomial and following usual procedures one finds a general solution

$$\theta(t) = C_1 * \cos\left(\sqrt{\frac{g}{L}} * t\right) + C_2 * \sin\left(\sqrt{\frac{g}{L}} * t\right) \qquad (2)$$

Approximations about y=0 up to order 3:



(order *n* approximation shown with *n* dots)

We will now build a physical pendulum and see how well our solution tracks the physical object. For this exercise the student needs to obtain a cord of about 36 inches (sewing thread will work) and a weight (something compact, like a fishing weight.) The author used a nut of 2.75 oz. Attach one end of the cord to the weight and secure the other end to a support that will let the cord and weight swing freely. Perhaps tie the thread to a push pin in the center of a door way. Carefully measure and record the length “L” of your pendulum. As you can see from the solution (2) the mass of the pendulum bob does appear which means that there is no compelling reason to accurately measure the mass. Anything compact and of two to three ounces will work. Decide on an initial displacement to start your pendulum. The $\sin(\theta) \approx \theta$ simplification works well for displacement up to about $15^0 = .2618$ radians.

You have a general solution for a pendulum, and now you have a particular pendulum that you have built.

Using <u>your pendulum</u> set your initial conditions (0) and $\theta'(0)$. With these initial conditions and the general solutionto the differential equation find your particular solution.

My pendulum has length 31.5 inches. The bob weighs 2.37 ounces, and my initial displacement was 14^0 = .2443 radians. You could measure an initial displacement with a protractor, an instrument which is no longer ubiquitous among students, (see illustration after REFERENCES below) or use the law of cosines to calculate the linear displacement of the bob from the rest position at 14^0.

$$d^2 = L^2 + L^2 - 2 \cdot L \cdot L \cos(\theta)$$

$$d^2 = 2L^2 - 2L^2 \cos(\theta)$$

$$d^2 = 2L^2 (1 - \cos(\theta))$$

$$d = L\sqrt{2(1-\cos(\theta))}$$

In my pendulum L = 31.5 inches and $=14^0$ so $d = 7\frac{11}{16}$. The particular solution for my pendulum is $\theta(t) = 0.2443 * \cos(\sqrt{\frac{g}{L}} * t)$. L is 31.5 inches and an accepted value for g in the units inches per second squared is g = 386.088 which makes my particular solution

$$\theta(t) = 0.2443 * \cos(\sqrt{\frac{386.088}{31.5}} * t) \tag{3}$$

Displace the bob and release it to start the pendulum swinging. Use a stop watch to measure the time for ten cycles. The period of your pendulum is the time required for one cycle and so the time value you measured above must be divided by ten. In this way I obtained an empirical estimate of the period (call it $\hat{P}$) of 1.792 seconds. To find the theoretical period of my pendulum I use my particular solution (3) and note that the period of $\cos(\sqrt{\frac{386.088}{31.5}} * t)$ is P=$2\pi\sqrt{\frac{L}{g}} = 1.794\ seconds$. This is good agreement with a percent error of -0.15%.

Let's do it again. This time my pendulum is 1.00965 meters long (I was trying to tie it off at 1 meter. I got close!) and I selected an initial displacement of 12^0 = .2094 radians. Solving for an initial displacement d as before we find d=.209 meters or 20.9 cm. And proceeding just as before, for my new pendulum I find a period $\hat{P} = 2.012$. With confidence in my measurements I assert that

$$\hat{P} = 2.012 \approx P = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.00965}{g}}$$

We can use this pendulum to solve for the acceleration due to gravity. Solving for g we find g=9.846 m/sec^2. We can use WolframAlpha to obtain a reverence value.

Using the value for local acceleration due to gravity as a reference we find our calculated value of g has a percent error of 0.56%. Again, this is good agreement.

REFERENCES

Edwards, C. Henry. 2008. *Elementary Differential Equations*. Upper Saddle River, NJ. Pearson/Prentice Hall.