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STUDENT VERSION
A COMPARISON OF NUMERICAL TECHNIQUES THROUGH THE STUDY OF THE NONLINEAR PENDULUM

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STATEMENT

When using a numerical method to model physical phenomena, we are often interested in observing the solution's behavior in the long run. Planetary motion and fluid flow are examples of applications where this might be important. If the numerical solution is "well-behaved" over time, we say it is *stable*. This idea can be made precise, but this informal notion will suffice for now. Of course we are also interested in accuracy and efficiency - i.e., we'd like to obtain a reasonable simulation in as few computations as possible, especially when pressed for computational resources.

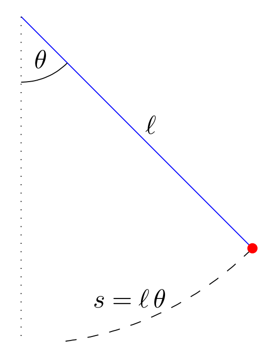

Figure 1. The simple pendulum.

Recall the simple pendulum of length ℓ as in Figure 1. From earlier we've seen that the differential equation to be solved is[1]

$$\theta'' = -\frac{g}{\ell}\sin(\theta),$$

where $g = 9.81$ m/s^2. Denoting the angular velocity as $\omega = \theta'$, write the resulting first-order system of ordinary differential equations:

$$\begin{aligned} \theta' &= \underline{\hspace{5cm}} \\ \omega' &= \underline{\hspace{5cm}}. \end{aligned} \tag{1}$$

Euler's method applied to the system (1) gives

$$\begin{aligned} \theta_{n+1} &= \underline{\hspace{7cm}} \\ \omega_{n+1} &= \underline{\hspace{7cm}}, \end{aligned} \tag{2}$$

We've seen that Euler's method is sometimes not adequate for modeling the pendulum realistically. Even when Δt was made very small, we observed the energy in the pendulum increases very quickly, leading to an unphysical simulation. The total energy of the pendulum computed in terms of our variables is given by[2]

$$\text{Total Energy} = \underbrace{\frac{1}{2}m(\ell\,\omega)^2}_{\text{Kinetic Energy}} + \underbrace{m\,g\,\ell(1-\cos(\theta))}_{\text{Potential Energy}}. \tag{3}$$

We will use this later when comparing other numerical solutions.

Around 1980, Abby Aspel (who at the time was a high school student) correctly coded up the Euler method for a related problem. Thinking the resulting inaccurate model was caused by an error in her code, she interchanged two lines in her program, and the model seemed to work. Abby accidentally stumbled upon the *semi-implicit* Euler method, given for our problem as follows:

$$\begin{aligned} \theta_{n+1} &= \theta_n + h\,\omega_n \\ \omega_{n+1} &= \omega_n + h\left(\frac{-g}{\ell}\sin(\theta_{n+1})\right), \end{aligned}$$

where as usual we use h as the step size Δt. Do you see the subtle difference between this and Euler's method in (2)? Briefly explain the difference in the space provided below.

It can be shown that the semi-implicit Euler method is in general only as accurate under Δt

[1]See the appendix for a derivation.
[2]See the appendix for a derivation.

refinement as Euler's method (that is, it is also a first-order method), so it will probably work just as badly - but we're open minded here, so let's test it out anyway.

We're certainly interested in accuracy - so let's also model (1) using the more accurate Improved Euler's method. Recall that the Improved Euler's method is *second-order* accurate. In the spaces provided below, write the Improved-Euler method for the pendulum problem.

$$\widetilde{\theta}_{n+1} = \underline{\hspace{8cm}}$$

$$\widetilde{\omega}_{n+1} = \underline{\hspace{8cm}}$$

$$\theta_{n+1} = \underline{\hspace{8cm}}$$

$$\omega_{n+1} = \underline{\hspace{8cm}}$$

It's time for you to test these out. We have provided files for your use. Make a copy of `EulerPendulum.m` and edit it to model the pendulum using the semi-implicit Euler Method, saving it with a different file name. Now do the same for the Improved Euler Method.

Test 1

To make our first experiment consistent, we will set $m = 1$ kg and $\ell = 1$ m, and release the mass from rest when the rod is horizontal to the ground - that is, the initial conditions are $\theta(0) = \pi/2$ and $\omega(0) = 0$. Using Euler's method, create a 100 second-long simulation of the pendulum using $\Delta t = 0.1$ (in other words, 1000 time steps), and plot your computed solution θ versus time. Then do the same for the other two methods. What do these solutions suggest about the physical behavior of the pendulum in each case? Now for each computed solution plot the total energy (see Equation (3)) as a function of time. Compare and discuss the behavior of the energy functions.

Test 2

Since the mass was released from rest, the true energy of the system in this case is equal to the initial potential energy: $|m\,g\,\ell| = 9.81$ Joules. Experimentally determine the largest possible step size for the semi-implicit Euler method to create a simulation whose energy stays within 1 Joule of the true energy. Now repeat, but this time use the Improved Euler Method. Measure the time required to compute the two optimal simulations.

Further Tests

Design your own experiment(s) to compare these methods. Play with the model - see how the methods perform with different parameter values, initial values, energy tolerances and/or step sizes.

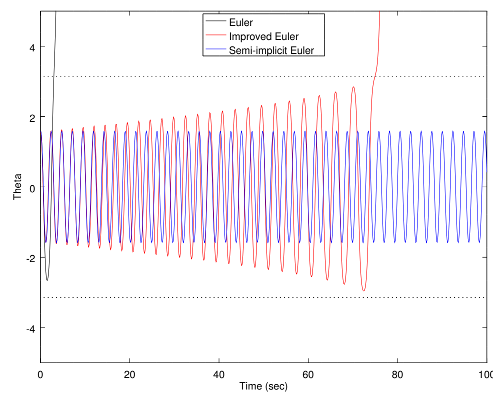

Figure 2. The computed solutions of the nonlinear pendulum problem using the Euler, Improved-Euler, and Semi-implicit Euler schemes. On the right, the y-axis have been altered to better inspect the behavior of the methods. The horizontal dotted lines indicate $\theta = \pm\pi$.

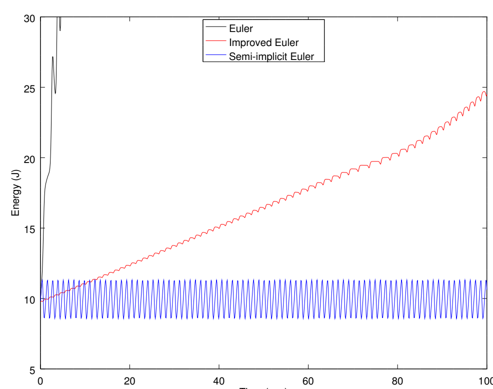

Figure 3. The calculated energy in the pendulum simulation using the Euler, Improved-Euler, and Semi-implicit Euler schemes. On the right, the y-axis have been altered to better inspect the energy behavior in the Improved Euler and Semi-implicit Euler methods.

You can even compare the methods on different planets by changing the value of gravity (say, a pendulum on Jupiter?).

Questions

1. Which of these three solvers do you recommend for this model? Fully justify your decision, including a few (not too many!) key examples you came across in your experiments. Summarize and present your results in a concise narrative, complete with diagrams and a conclusion.
2. Somewhere in your report, try to briefly explain (in layman's terms!) what is meant by the words "implicit" when describing numerical solvers. What sort of properties do implicit solvers tend to have? What are their pros/cons? Why do you think the Euler-Cromer method is called "semi-implicit"?

APPENDIX: Derivation of the Simple Pendulum

The simple pendulum consists of a mass m connected to the end of a rod of length ℓ whose other end is fixed to a frictionless hinge. Define θ as in Figure 4, where θ is measured in radians. The mass always moves along the circular arc $s = \ell\theta$, so the acceleration is given by $a = s''$. The component of gravity tangent to this arc is accelerating the mass along the arc of the circle. With some geometry and trigonometry we find that this acceleration is given by

$$a = -g\sin(\theta).$$

This gives

$$a = s'' = \ell\theta'' = -g\sin(\theta),$$

so the initial value problem to be solved is

$$\theta'' = -\frac{g}{\ell}\sin(\theta),$$

with $\theta(0)$ and $\theta'(0)$ specified.

Now we derive the energy formula in Equation (3) for the interested reader. From introductory physics, the total energy of a mass only being acted upon by gravity is given by

$$\text{Total Energy} = \underbrace{\frac{1}{2}mv^2}_{\text{Kinetic Energy}} + \underbrace{mgH,}_{\text{Potential Energy}} \tag{4}$$

where v is the velocity and H is the height of the mass relative to its lowest possible height (see, for example, [4, Ch. 8]). The velocity along the circular arc $s = \ell\theta$ is given by $v = (s)' = \ell\theta' = \ell\omega$, and from Figure 4 one can deduce that $H = \ell - \ell\cos(\theta)$. Substituting these into (4) yields (3).

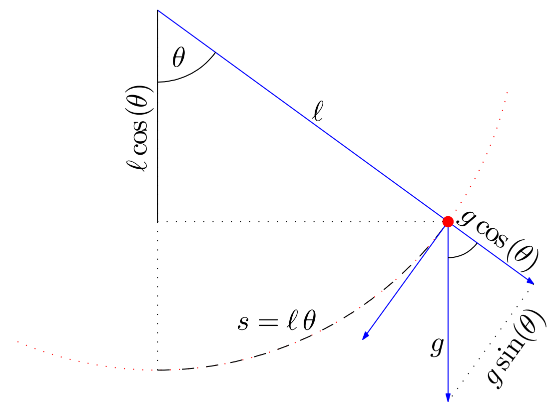

Figure 4. The simple pendulum with force diagram.

APPENDIX: Useful Code

We offer the following `MATLAB` files as supporting files:

- `EulerPendulum.m`
- `ImprovedEulerPendulum.m`

The Euler-Cromer version can be obtained by simply changing the `sin(th(j))` to `sin(th(j+1))` in line 21 of `EulerPendulum.m`.

REFERENCES

[1] Giordana, N. and H. Nakanishi. 2006. *Computational Physics*. Upper Saddle River, NJ: Pearson/Prentice.

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