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STUDENT VERSION

VARIABLE ASCENT RATE AND AIR MANAGEMENT IN SCUBA DIVING

AIR TO THE TOP

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STATEMENT

Persons who breathe compressed air will accumulate dissolved nitrogen in their body tissues. Transition from a high pressure environment to one of less pressure must be carefully controlled so as to reduce the chance of suffering one of the several kinds of decompression sickness. Recreational SCUBA divers are taught to ascend from depth slowly, giving dissolved nitrogen time to be eliminated in an orderly fashion through the breathing cycle. Everyone agrees that slow ascents are important practices for the health and safety of the diver, but the definition of "slow" has changed over time and between authorities. In this project we will look at ascent rates (which are derivatives) and calculate the amount of air required to make a safe ascent from various depths. One common rule taught to SCUBA divers is to ascend no faster than thirty feet per minute. The advantage of a rule that uses a constant ascent rate is that it is easy for a diver to remember and follow the rule. Dive computers on the other hand can pay attention to multiple parameters and can calculate a safe but variable ascent rate that depends on one's depth[1]. Dive computers are special purpose computers, waterproof and miniaturized, typically worn on the wrist. The dive computer constantly monitors the diver's environment measuring such things as current depth, maximum depth on this dive, breathing rate, remaining minutes on your air supply, no decompression limits and many more important pieces of information that most divers don't understand.

The issue with ascent rate is really an issue with reduction of ambient pressure. Ambient pressure (also referred to as absolute pressure) is the pressure due to water above the diver plus the pressure of one atmosphere of air above the water. The constant 33 appears in the formula for absolute pressure as a function of depth because the pressure of one atmosphere of air is equivalent to the pressure of 33 feet of sea water. In this paper we are assuming that diving is being conducted in sea water.

[1] The wisdom of variable ascent rates is under serious debate. There are at least two decompression theories: Reduced Gradient Bubble Method (RGBM) and Haldane Theory. Without getting into any details let it be said that according to Haldane Theory a quicker ascent at depth is advantageous.

At depth D feet the absolute pressure (in atmospheres) is $\frac{D+33}{33}$. Taking the ascent rule of 30 feet per minute as a starting point we can calculate that from a depth of 33 feet one can ascend to the surface in one minute and the change in absolute pressure will be from two atmospheres of absolute pressure (2 ata) to one atmosphere absolute (1 ata). This suggests that in one minute it is safe to ascend from a depth D_1 to a depth D_2 if the pressure ratio $\frac{P_1}{P_2} \leq 2$. In 1908 John Scott Haldane produced a set of decompression tables for the British Royal Navy based upon this principle. Haldane subjected goats to the pressure found at 165 feet sea water (fsw) and found that this pressure could be reduced by half without damage to the goats. At 165 fsw the pressure would be six atmospheres absolute absolute (6 ata = $\frac{165+33}{33}$ ata) and, Haldane found, this pressure could be reduced to 3 ata (or 66 ft) without an incident of decompression sickness. Empirical evidence suggests that this principle holds for other depth, pressure pairs. Decompression tables based upon Haldane's work were used by the Royal Navy until 1955. For these reasons we find it appropriate to start with the differential equation $P' = -\frac{1}{2}P \text{ and } P(0) = P_0$ where P' is the derivative of P with respect to time, t. Classify this differential equation. ___________________________

Solve the equation. Note $\frac{dP}{dt}$ is negative because we are ascending and as we ascend the pressure decreases. P(t) = _______________________.

As one is ascending it is easier and more intuitive to track one's depth and rate of change of depth rather than changes in pressure. Because of the relation between absolute pressure and depth we can substitute $\frac{D+33}{33}$ for P and $\frac{1}{33}D'$ for P' and write $D' = -\frac{1}{2}(D+33) \text{ and } D(0) = D_0$. The derivative is with respect to time and is negative because ascent is a <u>reduction</u> of depth. Classify this differential equation. ___________________________

Solve the equation. D(t) = _______________________.

Are these two results equivalent?

Your solution to (1) gives us a diver's depth at time t (in minutes) if he/she is ascending from an initial depth of D_0 and following the ascent rate[2] given in (1).

N.B. you may have classified the equation $D' = -\frac{1}{2}(D+33) \text{ and } D(0) = D_0$ as a variables separable initial value problem, and you would have been correct. But note the equation is algebraically equivalent to

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[2] It is assumed that this diver has no required decompression stops. A decompression stop is required if a diver has stayed under long enough to amass tissue pressure sufficient to violate the "reduction of .5 ambient pressure" rule on ascent. In this case a diver has a required decompression stop which means that ascent is stopped at a calculated depth for a period of time sufficient to allow the pressure in the body to reduce enough so that resuming ascent will not violate the "reduction of .5 ambient pressure" rule. This is called staged decompression and was addressed by Haldane.

$$\boldsymbol{D' + \frac{1}{2}D = 16.5} \qquad \textbf{(1)}$$

One recognizes this as a first order linear differential equation which is solved with the aid of the integrating factor $\boldsymbol{\rho(t)}$. Solve equation (1) as a linear equation. Are the results the same as your first solution?

Your solution to equation (1) should be

$$D(t) = (D_0 + 33)e^{-\frac{1}{2}t} - 33. \qquad (2)$$

Using our solution (2) we can calculate the safe ascent time from depth D_0 by setting D(t) = 0 and solving for t. Solve for t when D(t) = 0. You should find

$$t = -2[\ln(33) - \ln(D_0 + 33)] \qquad (3)$$

Verify this computation. ____

As an example, if a diver starts an ascent from a depth of, say 164 feet and maintains the ascent rate in (1) then using the results in (3) the ascent time will be? __________________

Divers, especially novice divers, have concerns about running out of breathing air. One somewhat popular piece of safety equipment is sold under the brand name "Spare Air." A Spare Air is a small cylinder containing an independent source of breathing air which could be employed if a diver's primary air supply is depleted or fails for some other reason. The Spare Air is available in 3.0 cubic foot and 1.7 cubic foot models. With the equations we have derived above we are now in a position to judge how much air would be required for a safe ascent from various depths and judge the adequacy of the air provided in a Spare Air. The point of the following calculations is find out how much air will be required to do a safe ascent, that is, an ascent that does not exceed the derived variable ascent rate expressed in equation (1).

For dive planning purposes a diver should know his or her surface consumption rate (SCR). The SCR is the diver's rate of consumption of air at or near the surface expressed in volume units/time unit. This is easily found by measuring air consumed while leisurely swimming about in a pool at a depth of say, ten feet for a time of say, ten minutes. One notes the pressure of air inside the SCUBA tank at the start of the ten minute period and upon completion the pressure is again noted. By itself the change in tank pressure does not tell you the amount of air consumed because not all SCUBA tanks are the same size. In the U.S. common size SCUBA tanks sizes are 63, 80 and 100 ft^3. These numbers reference the amount of air at 32 degrees Fahrenheit and one atmosphere pressure (standard temperature and pressure) that it takes to fill the tank to its working capacity (3000 psi for an aluminum tank.) The change in tank pressure is related to the air consumed by the formula "air consumed in ten minutes = pressure change * tank factor" where the "tank factor" is the amount of air in the full tank divided by the working pressure of the tank. (For a give tank material the working pressure is set by the U.S. Department of Transportation.) As an example, an 80 ft^3 aluminum tank will have a tank factor of 0.027. If a diver conducting the SCR test has a ten minute change in pressure of 190 psi then the SCR for that diver will be: $SCR = 190\, psi * \frac{\frac{0.027\, ft^3}{psi}}{10\, minutes} = 0.513\, \frac{ft^3}{min}$. Most divers will have SCR between .4 and .8 ft^3/min. The air consumed at depth D is a function of the depth and the SCR, specifically at depth D the consumption rate will be SCR $* \frac{D+33}{33}$ which is SCR multiplied by ambient pressure. Note that this is the RATE at which air is being consumed at depth D.

If we let D_0 be the starting depth and T^* be the time for a safe ascent calculated from (3) then the air consumed in this ascent will be

$$AC(D_0, T^*) = \int_0^{T^*} SCR * \frac{D(t)+33}{33}\, dt = \frac{SCR}{33}\int_0^{T^*} [(D_0 + 33)e^{-\frac{1}{2}t} - 33] + 33\, dt$$

$$= \frac{SCR}{33}(D_0 + 33)\int_0^{T^*} e^{-\frac{1}{2}t}\, dt = \frac{-2*SCR*(D_0+33)}{33}(e^{-\frac{1}{2}T^*} - 1) \qquad (4)$$

Verify these computations.

In this example, with D_0 = 164 ft we have seen that the safe time to surface is 3.5734 minutes so by (4) the air consumed will be 4.97 ft^3. The 3 ft^3 model Spare Air will supply 60% of the required air[3].

This work is not intended to be a critique of the Spare Air product but merely an example of mathematical applications to real situations. In an out of air emergency a diver is likely to abandon ascent rates and solve the most immediate problem by swimming rapidly to the surface in which case even 3 ft^3 of air will be most welcome. Note that an ascent without regard for ascent rates risk decompression sickness but in an out of air situation DCS is a secondary concern.

REFERENCES

Bennett, Peter. 1993. *The Physiology and Medicine of Diving*. Toronto: Harcourt Brace & Co.

Strauss, Michael B. and Igor V. Aksenov. 2004. *Diving Science*. Champaign, IL: Human Kinetics.

[3] Note that if one observed the constant 30 ft per minute rule for this ascent it would take 5.476 minutes and require 9.53 ft^3 of air.