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STUDENT VERSION

FISHERY HARVESTING: ATLANTIC COD

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STATEMENT

The logistic differential equation is often used to model the spread of disease as well as population growth. Can we use it to model sustainable harvesting of natural resources?

Fish are a valuable source of protein, and many people have lived to a large extent on fish and other seafood. Over fishing has driven many stocks of fish to near extinction [5, 6]. This applies in particular to the Mediterranean Sea, the Baltic Sea, and the North Atlantic Ocean. The collapse of the Newfoundland or Baltic sea cod fisheries should be taken as a severe warning that the fishery industry needs more careful controls [7]. With appropriate stock assessment data, mathematical models can be used to derive possible management strategies, which may aid the supervision of this industry.

U.S. stocks of Atlantic cod came close to commercial collapse in the mid-1990s. The 2012 assessments of Gulf of Maine and Georges Bank cod indicated both stocks are seriously over fished and are not recovering as quickly as expected. Based on these assessments, quotas for both stocks were significantly reduced in 2013 to help ensure over fishing does not occur and that these stocks rebuild. The Gulf of Maine cod quota was cut by 80% , and the Georges Bank cod quota was cut by 61%. National Oceanic and Atmospheric Administration (NOAA) Fisheries and the New England Fishery Management Council continue to work on management measures that will further protect cod stocks and provide opportunities for fishermen to target other healthy fish stocks instead of cod [10]. Now NOAA asks you to model the fish stock with harvesting in St. Georges Bank in order to fish sustainably.

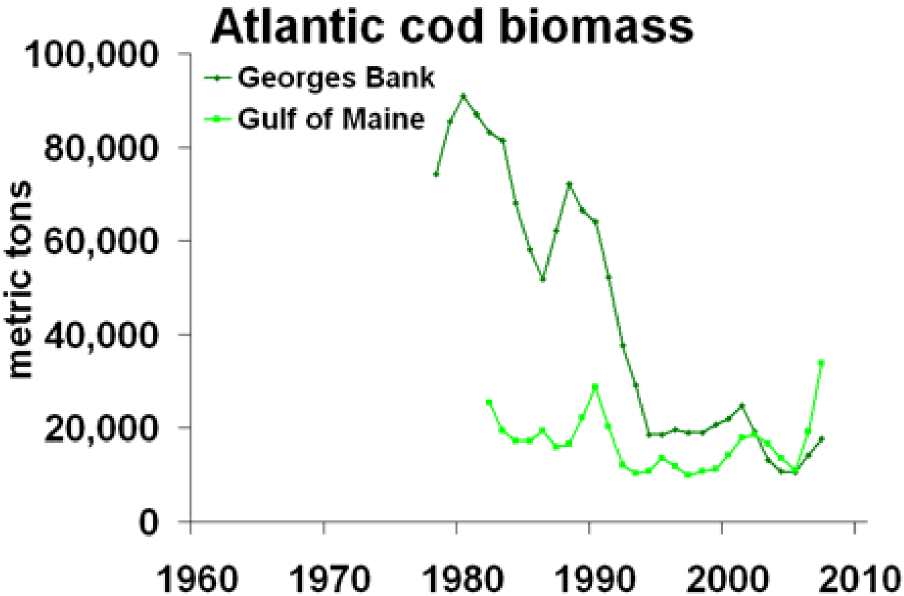

Figure 1. Atlantic Cod Biomass

Part I: Modeling Fish Harvesting

Let us consider the cod existing in a continuous, finite region with logistic growth and carrying capacity K, where the harvesting of cod is proportional to the stock available.

1. Define your variables and write down the differential equation for the Atlantic cod with harvesting proportional to the stock available. Suppose $u(t)$ is the population of cod at time t, r is the growth rate of the cod, h is the constant harvesting rate, u_0 is the initial cod population. Then a reasonable model [2] is given by:

$$\frac{du}{dt} = r\,u\left(1 - \frac{u}{K}\right) - h\,u, \quad 0 < t < T, \qquad (1)$$
$$u(0) = u_0,$$

   where T is the length of the fishing period.

2. Before solving the differential equation (1), what do you expect of the behavior of the fish stock? E.g., is the fish stock increasing or decreasing?
3. What kind of differential equation is this? Is it linear or nonlinear? What kind of method might you use to solve it?
4. Find the solution of the model. Atlantic cod biomass (in metric tons) and harvest rate in Georges Bank [11] are given in Table 1 from 1978 to 2008, could you estimate the parameters r and K from the data?
5. What is the long term behavior of the cod population? What does harvesting do for the cod?

Year	X_t	h_t	Year	X_t	h_t
1978	72,148	0.18847	1994	21,980	0.282701
1979	73,793	0.149741	1995	17,463	0.199275
1980	74,082	0.219209	1996	18,057	0.18781
1981	92,912	0.176781	1997	22,681	0.193574
1982	82,323	0.282033	1998	20,196	0.189526
1983	59,073	0.34528	1999	25,776	0.170108
1984	59,920	0.206545	2000	23,796	0.156601
1985	48,789	0.338185	2001	19,240	0.281787
1986	70,638	0.147236	2002	16,495	0.252869
1987	67,462	0.19757	2003	12,167	0.255417
1988	68,702	0.231541	2004	21,104	0.081034
1989	61,191	0.208597	2005	18,871	0.0873972
1990	49,599	0.335648	2006	21,241	0.0819517
1991	46,266	0.295344	2007	22,962	0.105181
1992	34,877	0.331848	2008	21,848	unknown
1993	28,827	0.350394	2009	-	-

Table 1. Annual (1978-2008) values of Atlantic cod biomass in metric tons, X_t, and harvest rate, $h(t)$, in metric tons in Georges Bank from [11].

Can we conserve the cod population? Using the data in Table 1, if we increase the constant harvest rate, h, to be 0.4, how will the population of Atlantic Cod change over time?

6. Constant harvesting may not be realistic. What other harvesting strategy can you think of? Address the issues (1) - (5) above for this harvesting strategy.

Part II: Challenge: Farm Fishing

There are several recent issues of *Natural Resource Modeling Journal* devoted to fishery management and we call attention to the survey paper [9]. The author traces the development of fisheries models from 1900 to the 21st century. He points out the fact that future fishery models will need to better address habitat and spatial concerns and to understand the effects of harvesting on the ecosystem. Managers have begun to use spatial management instruments, in the form of both permanent and temporary closures.

Assuming we implement fish farming for cod, so that cod live in a bounded region, for simplicity, we assume one-dimensional spatial domain with logistic growth and continuous diffusion [1]. We assume zero stock at the ends of the habitat, depicting a situation in which the habitat everywhere outside the domain in question cannot support the resource.

Let's think about how to add the spatial component. Hint to students: the term u_{xx} can describe the diffusion [1]. In the Appendix, you may find an intuitive way to derive and understand the diffusion partial differential equation. Note the students from Differential Equations II will be comfortable using u_{xx}, if this part is too challenging, you may skip questions 7-9.

7. We write down the new differential equation that includes the spatial component, let $u = u(x, t)$.

$$u_t = Du_{xx} + r\,u\left(1 - \frac{u}{K}\right) - h\,u, \quad 0 < x < l,\ 0 < t < T \tag{2}$$

   What kind of initial and/or boundary conditions do we need to solve this problem? Hint: We need to specify both initial condition $u(x, 0) = f(x)$ and boundary conditions $u(0, t) = g_1(t)$ and $u(l, t) = g_2(t)$ (in our case, we have $g_1(t) = g_2(t) = 0$ since we have hostile environment). Interpret these conditions in this fishery setting. D is the diffusion rate.

8. How might a solution strategy differ from Part I used in solving (1)? What do you anticipate from your model by adding the spatial component?

9. **Ponder:** For an even more realistic situation, suppose the harvesting h depends on the spatial location and the time. Write out a differential equation which might model this phenomenon. What is the difference between this newer result and your result from Question 8?

Acknowledgement

We used a simplified version of the models from W. Ding, G.E. Herrera, H.R. Joshi, S. Lenhart and M.G. Neubert [3, 4].

Appendix: Diffusion

Description of movement arising from many short movements in random directions. Suppose an organism moves along a line: moving a distance Δx to the left with probability $\frac{1}{2}$, moving a distance Δx to the right with probability $\frac{1}{2}$. Suppose $p(x, t) =$ probability that the organism is at location x at time t, then

$$p(x, t) = \frac{1}{2}p(x + \Delta x, t - \Delta t) + \frac{1}{2}p(x - \Delta x, t - \Delta t). \tag{3}$$

Subtracting $p(x, t - \Delta t)$ from both sides and divided by Δt, we have

$$\frac{p(x, t) - p(x, t - \Delta t)}{\Delta t} = \frac{p(x + \Delta x, t - \Delta t) - 2p(x, t - \Delta t) + p(x - \Delta x, t - \Delta t)}{2\Delta t}. \tag{4}$$

Imposing the scaling $\frac{(\Delta x)^2}{\Delta t} = 2D$, we obtain

$$\frac{p(x, t) - p(x, t - \Delta t)}{\Delta t} = \frac{D\Big[p(x + \Delta x, t - \Delta t) - 2p(x, t - \Delta t) + p(x - \Delta x, t - \Delta t)\Big]}{(\Delta x)^2}. \tag{5}$$

As $\Delta x, \Delta t$ go to 0, $\dfrac{p(x, t) - p(x, t - \Delta t)}{\Delta t}$ goes to $p_t(x, t)$, and

$$\frac{\left[p(x+\Delta x, t-\Delta t) - 2p(x, t-\Delta t) + p(x-\Delta x, t-\Delta t)\right]}{(\Delta x)^2}$$

goes to $p_{xx}(x,t)$. So we have the diffusion equation

$$p_t = Dp_{xx}.$$

REFERENCES

[1] Cantrell, R.S. and C. Cosner. 2003. *Spatial Ecology via Reaction-Diffusion Equations*. Chichester, West Sussex, UK: John Wiley & Sons.

[2] Clark, C. W. 1990. *Mathematical bioeconomics: The optimal management of renewable resources, 2nd edition*. New York: John Wiley & Sons.

[3] Ding, W. and S. Lenhart. 2009. Optimal Harvesting of a Spatially Explicit Fishery Model. *Natural Resource Modeling*. 22:(2): 173-211.

[4] Joshi, H.R., G.E. Herrera, S. Lenhart, and M.G. Neubert. 2009 Optimal Dynamic Harvest of a Mobile Renewable Resource. *Natural Resource Modeling*. 22(2): 322-343.

[5] Hilborn, R. and C. J. Walters. 1992. *Quantitative Fisheries Stock Assessment: Choice, Dynamics and Uncertainty*. London: Chapman and Hall, Inc.

[6] Hilborn, R. 2012. *Overfishing: What Everyone Needs to Know*. Oxford: Oxford University Press.

[7] Kurlansky, M. 1998. *Cod: A biography of the fish that changed the world*. London: Penguin Books.

[8] Nolen, J. 2009. *Partial Differential Equations and Diffusion Processes*. `http://math.stanford.edu/~ryzhik/STANFORD/STANF227-12/notes227-09.pdf`. Accessed 8 December 2015.

[9] Quinn, T.J. II. 2003. Ruminations on the Development and Future of Population Dynamics Models in Fisheries. *Natural Resource Modeling*. 16(4): 341-392, 2003.

[10] Risky Decisions: How denial and delay brought disaster to New England's historic fishing grounds: A brief from The PEW Charitable Trusts. October, 2014. `http://www.pewtrusts.org/~/media/assets/2014/09/risky-decisions-brief_final.pdf?la=en`. Accessed 8 December 2015.

[11] Yakubu, A.A., N. Li, J.M. Conrad, and M-L. Zeeman. 2011. Constant proportion harvest policies: dynamic implications in the Pacific halibut and Atlantic cod fisheries. *Mathematical Biosciences*. 232: 66-77.