www.simiode.org
SIMIODE Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations

STUDENT VERSION
SPRING MASS DAMPED

Keith Landry
Civil Engineering and
Construction Management
Georgia Southern University
Statesboro GA USA

Brian Winkel
SIMIODE
26 Broadway
Cornwall NY USA

STATEMENT

Consider a mass suspended by a spring, as depicted in Figure 1. If we let the mass hang still, thus extending the spring from its natural length, we will see that the mass comes to rest at what is called *static equilibrium*. In attempting to model the vertical motion of this mass we impose a coordinate system for $y(t)$, the vertical displacement of the mass from the static equilibrium. Engineers refer to such a system as a Single Degree of Freedom System (SDOFS), as we are tracking only one variable, namely, $y(t)$, vertical displacement from the static equilibrium. For consistency, let us say $y = 0$ is the spring's vertical displacement at the spring's static equilibrium and is the distance of the displacement of the mass from that static equilibrium and denote positive in the downward direction and negative in the upward direction. That is, if we extend the spring 3 cm downward then $y = 3$, while if we compress the spring upward 2 cm then $y = -2$.

We are going to use a Free Body Diagram (Figure 2) to depict all the vertical forces acting on the mass in this case. Figure 2 depicts the mass pulled down just a bit beyond the static equilibrium position and moving downward, with the two forces acting on it being (1) the restoring force of the spring which acts in an upward vertical direction and (2) the force upward from the damping due to friction in the spring and the resistance of the media on the moving mass.

Remember, Newton's Second Law of Motion says that for a given body the mass of that body times the acceleration of that mass is equal to the sum of the external forces acting on the mass. In the undamped scenario [1], we extended the mass downward from it's initial position of static equilibrium by a distance $y(t)$. Since the spring mass system was at rest, the downward force due

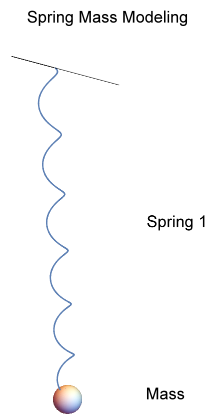

Figure 1. Simple spring mass configuration.

to gravity, $m \cdot g$ (mass m times acceleration due to gravity g) cancelled out the restoring force of the spring. If we now imagine releasing the mass, we can envision a bobbing motion will ensue. If we took a snapshot of the mass as it moves downward while bobbing, our Free Body Diagram in Figure 2 would then show two forces acting on the mass, (1) the restoring force of the extended spring and (2) the combined resistance of the spring and media on the moving mass which serves to dampen out the motion in time; both forces in the upward direction.

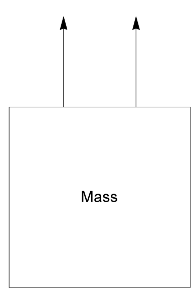

Figure 2. Template for Free Body Diagram.

Let us examine (1) the restoring force first. In high school you may have conducted an experiment on springs to verify that when a spring is extended a distance y beyond its equilibrium then there is a force of restoration to the static equilibrium position, $F = k \cdot y$, acting to restore the spring to its equilibrium and that force is proportional to the displacement only. The constant of proportionality, k, is called the *spring constant* and it can be determined experimentally by suspending different masses on the spring, each mass producing a corresponding force. Recall $F = m \cdot g$, where g is the acceleration due to gravity. If we denote the corresponding displacement y, then when plotting F vs. y the data is linear and goes through the origin, as 0 units of force displaces the spring 0 units of distance. k can thus be determined by fitting the line $F = k \cdot y$ to the data F vs. y for a given spring. This equation is called Hooke's Law and is named after 17^{th} century British physicist Robert Hooke.

Direction is important here, for if the spring is extended (hence the mass is down below its static equilibrium) then the restoring force of the spring force is in the opposite (up) direction and if forces and position are positive when downward then this restorative force must be negative.

A spring is thus said to have *stiffness* which we can characterize as a force proportional to its extension from static equilibrium, namely, $F = k \cdot y$.

The damping or resistance force (2) is what slows down the oscillation, eventually causing the mass to come to rest at the static equilibrium. This damping occurs due to the friction in the spring and the resistance of the medium (in our case air) to the motion of the mass itself. A reasonable term for such damping is $c \cdot y'(t)$, i.e. the faster the mass moves the more resistance is offered. Direction is important here, for if the mass is moving downward then the damping force is in the opposite (up) direction and if forces and position are positive when downward then the damping force must be negative.

Modeling Activities

1. Identify the forces acting on the mass shown in Figure 2 and if y is positive and the spring is extended downward then in what direction are these forces? If y is negative and the spring is compressed upward then in what direction is this force? If the spring is moving up then in what direction is the resistance force?
2. Complete the differential equation (1) using these forces in Newton's Second Law of Motion (again, here m is the mass):

$$m \cdot y''(t) = \underline{\hspace{6cm}} \tag{1}$$

3. Write out an initial value problem using (1) if the spring's initial displacement from static equilibrium is y_0 and its initial velocity is v_0.

You now have a mathematical model of a mass bobbing up and down on a spring in which there is a force due to resistance to the motion, i.e. damping.

4. If we believe our model with damping will oscillate up AND dampen then what kind of function(s) does that suggest for our vertical displacement of the mass, $y(t)$? Why, some kind of sine or cosine function, BUT dampened, perhaps something like $y(t) = Ae^{-\lambda t}\sin(\omega t)$ or $y(t) = Be^{-\lambda t}\cos(\omega t)$. But if we were to use the sine function term only, then since it always has $y(0) = 0$ we could never model a mass with a non-zero displacement, so we would also need the cosine function in our solution. Similarly, we could make an argument for not using just the sine term. So why not have a little of each, i.e. a of $e^{-\lambda t}\sin(\omega t)$ and b of $e^{-\lambda t}\cos(\omega t)$? Thus

our candidate or conjectured solution for the vertical displacement from static equilibrium for our mass on a spring without resistance might look like

$$y(t) = a\, e^{-\lambda t} \sin(\omega t) + b\, e^{-\lambda t} \cos(\omega t)\,. \tag{2}$$

One way to tell how good this conjecture is would be to just try it out “for size” by substituting it into your differential equation model you built from (1). Go ahead and do that to see where it leads. Remember you are trying to figure out if (2) is a solution and then if it is a solution what might a, b, and ω have to be to fulfill the initial conditions $y(0) = y_0$ and $y'(0) = v_0$?

5. What information does k, m, and c give you in your confirmed conjectured solution, say about the possible values of ω and λ? Explain. WARNING: The algebra and differentiation, as well as the equation solving, can get very complicated; so doing these calculations by hand is not suggested. A computer algebra system such as Maple or Mathematica is strongly suggested.
6. Let us take an actual spring with a mass and use your model with some initial conditions to find a complete solution. Use $m = 7$ gram, stiffness coefficient or spring constant $k = 40$ dyne/cm, resistance or damping coefficient $c = 5$ dyne/(cm/s), with initial displacement (downward) $y(0) = 1$ cm and initial velocity (downward) $y'(0) = 2$ cm/sec. Here units are in grams, centimeters, seconds, and dynes appropriately. Solve the differential equation in the above manner and graph its motion over a reasonable interval of time. Explain what you see.
7. Vary the values for m, c, and k and explain what you see in each case.
8. If you have access to a computer algebra system then solve the differential equation with the appropriate initial conditions and compare the solution you obtain in this way to your solution from the method above.

Supporting Theory

There is supporting theory (called existence and uniqueness theorems) in mathematics which says that there will be a solution (existence) and if we find a candidate solution for our differential equations which also satisfies the initial conditions then this solution is the unique solution (uniqueness) and we need look no further. Thus our intuition has helped us to find the unique solution. We shall develop other means of finding “candidate” solutions in our further studies.

REFERENCES

[1] Landry, K. A. and B. Winkel. 2016. 3-101-S-SpringMassFirstTry. `https://www.simiode.org/resources/2457`. Accessed 22 July 2016.