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SIMIODE Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations

STUDENT VERSION
STOCHASTIC POPULATION MODEL

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STATEMENT

We will develop stochastic or probabilistic models; for the probability that a population is size N at time t. In actual populations there is a random component to the growth that may not be captured in simpler differential equations models. We examine this approach in a straightforward way.

We are going to describe the population by an infinite sequence of functions:

$$P_0(t), P_1(t), \ldots, P_N(t), \ldots . \tag{1}$$

- Here, $P_N(t)$ is the probability that at time t the population size is N.
- At a given time t we have a probability distribution: $0 \leq P_N(t) \leq 1$ for $N = 0, 1, 2, \ldots$, and $\sum_{j=0}^{\infty} P_j(t) = 1$.
- As time t moves forward the probability distribution evolves. If the population is probabilistically growing, the bulk of the probabilities is moving to $P_N(t)$ for larger and larger N.

Assumptions

1. We are going to allow births, but not deaths just to make our modeling simpler.
2. During a very short time interval the only two possibilities for the population are to (1) remain constant or (2) go up by 1 (single birth). This assumption does not apply for long time intervals.
3. There is a positive constant r (a birth rate) such that for very short time intervals Δt in a population of size N the probabilities of population growth are given approximately as follows:

$$\begin{aligned} \text{Prob(1 birth)} &\approx rN(\Delta t) \\ \text{Prob(0 births)} &\approx 1 - rN(\Delta t) \end{aligned}$$

So for short time periods the probability of a birth is proportional to the time elapsed and the size of the population (number of possible parents). You can think of $N\Delta t$ as having units of "man-hours" (or perhaps more appropriately "woman-hours").

The dynamical system

We develop the differential equation models for each probability distribution $P_N(t)$':

$$\begin{aligned} P_N(t+\Delta t) &= P_{N-1}(t) \times \text{Prob}(1 \text{ birth between } t \text{ and } t+\Delta t) \\ &\quad + P_N(t) \times \text{Prob}(0 \text{ births between } t \text{ and } t+\Delta t) \\ &= r(N-1)\Delta t P_{N-1}(t) + (1 - rN\Delta t)P_N(t) \end{aligned}$$

Thus

$$\frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} = r(N-1)P_{N-1}(t) - rNP_N(t)$$

and in the limit

$$\frac{dP_N(t)}{dt} = r(N-1)P_{N-1}(t) - rNP_N(t)\,.$$

We also have initial conditions corresponding to a definite starting population of N_0:

$$P_0(0) = \cdots = P_{N_0-1}(0) = 0; \quad P_{N_0}(0) = 1; \quad P_{N_0+1}(0) = \cdots = 0.$$

One can think of this as a compartment model, where the bins correspond to possible populations with probability flows from one bin to the next.

ACTIVITIES

Activity 1 - Model Building

a) What is the difference equation (and limiting differential equation) for $P_0(t)$?

b) Solve the differential equation in (a) for $P_0(t)$.

c) Use the differential equation for $P_0(t)$ in the differential equation you can obtain for $P_1(t)$ and solve for $P_1(t)$.

d) Continue to develop the solutions for $P_2(t)$, $P_3(t), \ldots, P_N(t)$ in the same way.

e) Confirm the solutions for $P_N(t)$, $N = 0, 1, 2, \ldots$ satisfy the differential equations in which they appear.

Activity 2 - Expected Value

Since the model is stochastic, we can not give the future population precisely. We can give statistics on the future population. For instance, we can compute the mean or average population size at time t. To do this we use the notion of *expected value* defined below.

The *expected value* of population $N(t)$ at time t with initial population $N(0) = N_0$ is,

$$E(N(t)) = \sum_{n=0}^{\infty} n \cdot P_n(t)\,, \tag{2}$$

a) Determine $E(N(t))$. Hint: First develop a differential equation for $E(N(t))$. (Alternatively, use the explicit formulas for $P_n(t)$ found in Activity 1.)

b) How does the average population at time t compare to the results we would obtain from a deterministic model in which we posit that $N'(t) = rN(t)$ with initial condition $N(0) = N_0$?

Activity 3 - Variance

With more work you can find the variance:

$$\mathrm{Var}(N(t)) = E((N(t) - E(N(t))^2) = N_0(1 - e^{-rt})e^{2rt}.$$

Standard deviation increases with time, so future predictions are made with less and less certainty.