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STUDENT VERSION
TUNED MASS DAMPERS - PART II

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STATEMENT

We refer the reader to Part I of this Modeling Scenario [9] for background material and rationale for engineers to use Tuned Mass Dampers. A very good explanation of just what a Tuned Mass Damper (TMD) is can be seen in the YouTube video at [16] where the right claim is that TMDs offer an example of elegance in engineering. Basically, a TMD is a pair of coupled damped harmonic oscillators. The name comes from their use in stabilizing things like buildings or bridges or machinery which can be forced to oscillate by the wind or other forces.

Modeling a Tuned Mass Damper

We will continue (from Part I[9]) to study the oscillation of a mass structure. We represent the stiffness of the original mass of a structure as depicted in Figure 1 using a spring and the energy absorption from that mass using a damper. We attach a spring and damper (TMD) to the original mass and explore how much this TMD can reduce vibrations due to external forces. Displacement of the mass will be measured by distance from an initial equilibrium position with a single degree-of-freedom. Displacement of the TMD mass will be measured by distance from an initial equilibrium position with a second degree-of-freedom about its own initial equilibrium position.

Consider the following situation shown in Figure 1. A large mass m_1 on a "spring" with spring constant k_1 and a resistance force or damper proportional to velocity of the mass with damping coefficient c_1 is coupled to a smaller mass m_2 by a spring with spring constant k_2 and damper with damping coefficient c_2. We will assume $c_1 \neq 0$ and $c_2 \neq 0$, i.e. we consider the more realistic (than seen in Part I[9]) situation in which both original structure and damper experience damping.

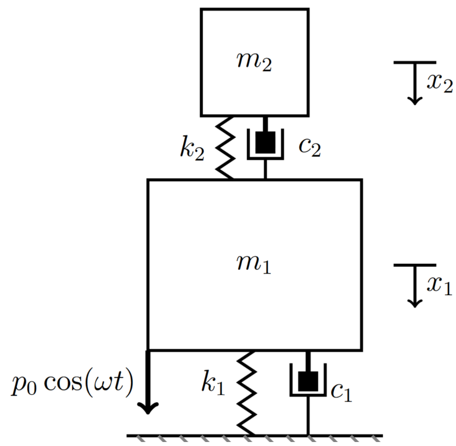

Figure 1. Diagram of the TMD for developing full set of two degrees of freedom differential equation model [11].

Recall in Part I [9] that we assumed $c_1 = 0$ and $c_2 = 0$ so we had no damping of the original structure or our attached damper. Thus our original structure could experience resonance without a damper attached. Now in Part II we first study the situation in which the original structure has resistance to sway or damping, $c_1 \neq 0$, but still no attached TMD. We will see that with damping ($c_1 \neq 0$) the displacement of the original structure reaches a maximum amplitude; however, it does not go into resonance. This maximum amplitude could be dangerous and so we want to employ a TMD to reduce this possibly too high an amplitude of the structure's displacement. Indeed, we want to confirm in this case that by tuning the values of m_2 and k_2 the maximum amplitude of m_1's oscillations can be lowered considerably. However, we first offer a review of the undamped structure.

Structure Without Resistance or Damping - Review from Part I

In this case with no resistance ($c_1 = 0$ and $c_2 = 0$) when we apply an external load or driver force, $f(t)$, we have the following single degree of freedom ($y(t)$) differential equation model for the displacement of our mass from its static equilibrium, i.e. a spring-mass (no dashpot) system:

$$my''(t) + ky(t) = f(t)\,, \quad y(0) = y_0\,, \quad y'(0) = v_0. \tag{1}$$

The natural frequency of mass m_1 can be obtained from the solution of the homogeneous portion, (2), of our differential equation (1),

$$my''(t) + ky(t) = 0, \quad y(0) = y_0\,, \quad y'(0) = v_0. \tag{2}$$

From (3) we see that the natural frequency of the undriven or homogeneous solution of (2) is $\omega_0 = \sqrt{\frac{k}{m}}$ while the complete homogeneous solution to (1) is:

$$y(t) = \frac{v_0\sqrt{m}\sin\left(\sqrt{\frac{k}{m}}t\right) + y_0\sqrt{k}\cos\left(\sqrt{\frac{k}{m}}t\right)}{\sqrt{k}}\,. \tag{3}$$

Recall what happens if the driver function $f(t)$ has the same frequency as the natural frequency, ω_0, of the spring-mass system - RESONANCE and trouble. To see this we divide both sides of (1) by m and rearrange the resulting equation using $\omega_0^2 = \frac{k}{m}$ to obtain (4):

$$y''(t) + \omega_0^2 y(t) = \frac{1}{m}f(t)\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{4}$$

Now if $f(t) = \sin(\omega t)$ and this driver has the same natural frequency as our mass system, i.e. $\omega = \omega_0$, then our solution of (4) is seen in (5) with a plot offered in Figure 2.

$$y(t) = -\frac{t\cos(\omega_0 t)}{2m\omega_0} + \frac{\sin(\omega_0 t)}{2m\omega_0^2} - \frac{\sin(\omega_0 t)\cos^2(\omega_0 t)}{2m\omega_0^2} + \frac{\sin(2\omega_0 t)\cos(\omega_0 t)}{4m\omega_0^2} + \frac{v_0\sin(\omega_0 t)}{\omega_0} + y_0\cos(\omega_0 t)\,. \tag{5}$$

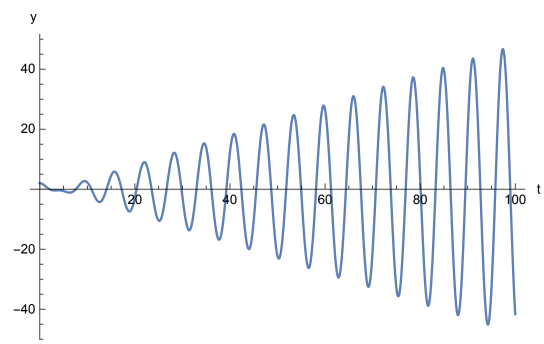

Figure 2. Plot of solution of (4) with values, $\omega_0 = 1$, for the natural frequency of the system and the driver as well, and $y_0 = 2, v_0 = 0$, and $m = 1$. In this case $f(t) = \sin(\omega t)$ is the driver function.

The key term in this solution is the very first term. All the other terms are of period $\frac{2\pi}{m}$ with finite amplitudes. The first term $-\frac{t\cos(\omega_0 t)}{2\omega_0}$ grows without bound and accordingly we refer to this solution phenomenon as *resonance*. For the following values, $\omega_0 = 1, y_0 = 2, v_0 = 0$, and $m = 1$ we plot our solution (5) over the time interval $[0, 100]$ in Figure 2 and see the solution grows indefinitely; this is resonance.

Structure With Resistance or Damping

In Part I [9] and in the previous section we analyzed the case where there is no resistance or damping in either the original structure or the damper, i.e. $c_1 = c_2 = 0$, and we found that if we tuned the damper by letting its natural frequency be the same as the natural frequency of the original structure we would stop the resonance and effectively reduce the oscillation of the original structure to 0 when a driving force with the same frequency as the natural frequency of the structure occurred. This means if we prescribe k_2 and m_2 so that $\sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}}$ in the model (6) for the original structure's displacement, $x_1(t)$, and the TMD's displacement, $x_2(t)$ we will have no resonance, indeed, no motion of the original structure when it is driven by an external force $p_0\cos(\omega t)$ with $\omega = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}}$.

$$\begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) - k_2(x_1(t) - x_2(t)) + p_0\cos(\omega t) \\ m_2 x_2''(t) &= -k_2(x_2(t) - x_1(t)) \end{aligned} \tag{6}$$

We now permit resistance or a damping force on our original mass, m_1. So we apply a nonzero resistance or damping force $c_1 x_1'(t)$ to the forces acting on mass m_1, but allow no TMD to be applied. That is, in this case we have resistance ($c_1 \neq 0$ and $c_2 = 0$) and no attached damper when we apply an external driver force, $f(t)$. Indeed, we have just the following single degree of freedom ($y(t) = x_1(t)$) differential equation model for the displacement of our mass, m_1, from its static equilibrium:

$$my''(t) + cy'(t) + ky(t) = f(t)\,, \quad y(0) = y_0\,, \quad y'(0) = v_0. \tag{7}$$

If we use $f(t) = p_0\cos(wt)$ as a driver then the closed form solution of (7) is given by (8):

$$\begin{aligned} y(t) = \; & \frac{m_1 p_0 \omega^2 \exp\left(\frac{t\sqrt{c_1^2-4k_1m_1}}{m_1} - \frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}\right)}{2\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} - \frac{c_1 m_1 p_0 \omega^2 \exp\left(\frac{t\sqrt{c_1^2-4k_1m_1}}{m_1} - \frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}\right)}{2\sqrt{c_1^2-4k_1m_1}\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} \\ & - \frac{k_1 p_0 \exp\left(\frac{t\sqrt{c_1^2-4k_1m_1}}{m_1} - \frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}\right)}{2\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} - \frac{c_1 k_1 p_0 \exp\left(\frac{t\sqrt{c_1^2-4k_1m_1}}{m_1} - \frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}\right)}{2\sqrt{c_1^2-4k_1m_1}\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} \\ & + \frac{m_1 p_0 \omega^2 e^{-\frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}}}{2\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} + \frac{c_1 m_1 p_0 \omega^2 e^{-\frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}}}{2\sqrt{c_1^2-4k_1m_1}\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} - \frac{k_1 p_0 e^{-\frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}}}{2\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} \\ & + \frac{c_1 k_1 p_0 e^{-\frac{t\left(\sqrt{c_1^2-4k_1m_1}+c_1\right)}{2m_1}}}{2\sqrt{c_1^2-4k_1m_1}\left(c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4\right)} + \frac{c_1 p_0 \omega \sin(t\omega)}{c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4} - \frac{m_1 p_0 \omega^2 \cos(t\omega)}{c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4} \\ & + \frac{k_1 p_0 \cos(t\omega)}{c_1^2\omega^2 + k_1^2 - 2k_1m_1\omega^2 + m_1^2\omega^4}\,. \end{aligned} \tag{8}$$

We will look for the steady state response of mass m_1 to the driver and thus ignore the decaying exponential terms of (8). This means we are interested only in the pure cosine and sine terms. We use the trigonometric identity (9) which is often used by engineers to combine two periodic functions with the same frequency into one phase-shifted periodic function.

$$A\cos(x) + B\sin(x) = \sqrt{A^2+B^2}\sin\left(x + \arctan\left(\frac{A}{B}\right)\right) \tag{9}$$

for $B \neq 0$ and $-\frac{\pi}{2} < \arctan\left(\frac{A}{B}\right) < \frac{\pi}{2}$ to determine the resulting amplitude, $\sqrt{A^2+B^2}$ of the steady state solution here, i.e. when $x = wt$. First we determine the coefficients of $\cos(\omega t)$ (A) and of $\sin(\omega t)$ (B) from which we can compute the magnitude, $\sqrt{A^2+B^2}$, of the steady state solution.

We find

$$A = \frac{k_1 p_0}{c_1^2\omega^2 + k_1^2 - 2k_1 m_1 \omega^2 + m_1^2\omega^4} - \frac{m_1 p_0 \omega^2}{c_1^2\omega^2 + k_1^2 - 2k_1 m_1 \omega^2 + m_1^2\omega^4} \tag{10}$$

and

$$B = \frac{c_1 p_0 \omega}{c_1^2\omega^2 + k_1^2 - 2k_1 m_1 \omega^2 + m_1^2\omega^4} \tag{11}$$

from which we (with algebraic simplification) can compute the amplitude of the steady state response of mass m_1 to the driver function $f(t) = p_0 \cos(\omega t)$, after simplifying:

$$\text{Amp}(\omega) = \sqrt{\frac{p_0^2}{c_1^2\omega^2 + k_1^2 - 2k_1 m_1 \omega^2 + m_1^2\omega^4}} \tag{12}$$

From $\text{Amp}(\omega)$, the amplitude of the steady state response of mass m_1, we can determine the frequency, ω_M, at which the displacement or amplitude of position $x_1(t)$ is greatest. We shall refer to this as the *maximum frequency response* of mass m_1.

We show a typical maximum frequency response curve in Figure 3.

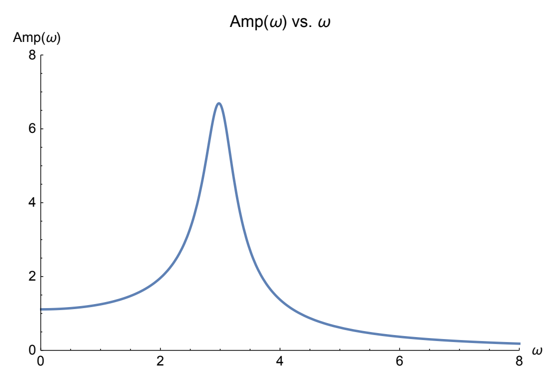

Figure 3. Maximum frequency response plot for the steady state solution of (7) with driver $f(t) = p_0 \cos(\omega t)$ for $p_0 = 100$, $y_0 = 0$, and $y'(0) = v_0 = 0$, while $m = 10$, $k = 90$, and $c = 5$.

The maximum frequency response of mass m_1 can be obtained by differentiating $\text{Amp}(\omega)$ with respect to ω and setting that derivative equal to 0 to find the maximum. With the numbers offered in Figure 3 using this optimization process we find that the maximum amplitude response of 6.68994 occurs at $\omega = 2.97909$.

So while we do not have resonance, which we can obtain when $c = 0$ and the driver frequency is the same as the structure's natural frequency, we do have a frequency, $\omega = 2.97909$, which is close to that of the natural frequency of the undamped structure, namely $\omega = 3$, wherein if the structure possessed damping ($c_1 \neq 0$) there would be a maximum frequency response which could be unacceptable. Hence we seek the characteristics of a TMD which will mitigate this maximum frequency response in the more realistic case.

Large Frequency Amplitude Responses Spell Trouble

In general, engineers do not like resonance. Neither do they like a large amplitude response to possible driver frequencies, as it means things could be getting out of control, going beyond tolerance levels, and possibly causing havoc. For suppose (7) were a model of a tall structure of mass m, k represented a restorative force or stiffness in the structure, and c represented a resistance or damping in the sway of the structure due to friction or internal resistance to motion in the structure. If $y(t)$ is the horizontal displacement of the corner of the structure from its static equilibrium due to a force, $f(t)$, from seismic tremors or wind then a high amplitude response to these forces would mean that the structure would swing broadly and could eventually break. Thus engineers design in light of amplitude frequency responses to typical driver frequencies. In particular, if the structure has certain properties, namely, m_1, k_1, and c_1, then the structure needs to be able to counter (better that than just sustain) the force $f(t) = p_0 \cos(wt)$ and here is where a TMD can mitigate the maximum amplitude frequency response. So how do we tune the TMD in this situation?

Activities

Activity 1

Using Newton's Second Law, which says that the mass times the acceleration of that mass is equal to the sum of all external forces acting on the mass, and Figure 1 as a Free Body Diagram source, show that the governing equations for the motions of the structure and the damper are given by (13). Hint: For consistency all downward forces, positions, velocities, and accelerations are positive.

$$\begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) - c_1 x_1'(t) - k_2(x_1(t) - x_2(t)) - c_2(x_1'(t) - x_2'(t)) + p_0 \cos(\omega t) \\ m_2 x_2''(t) &= -k_2(x_2(t) - x_1(t)) - c_2(x_2'(t) - x_1'(t)) \end{aligned} \qquad (13)$$

In our study here we will now presume an "ideal" structure with no resistance or damping coefficient, i.e. $c_1 \neq 0$, and the same for the damper, i.e. $c_2 \neq 0$. As we can see there is a driving force applied to the structure of $f(t) = p_0 \cos(\omega t)$. Our interest will be in seeing if we can tune the added mass damper to reduce or eliminate the possibility of large maximum amplitude frequency responses when the initial structure is driven by a force with a frequency equal to that of the natural frequency of the initial mass system.

Activity 2

a) Determine the form of (13) when $c_1 = 5$ and $c_2 = 0.5$.

b) In the resulting system use $m_1 = 10$ and $k_1 = 90$ with $y(0) = 0$ and $y'(0) = 0$, while $f(t) = 10\cos(3t)$ and set $m_2 = 0$ with $k_2 = 0$, i.e. withdraw the second mass system. Explain the physical significance and impact results of the numbers 10, 90, and 3 in the above sentence in terms of the motion of the initial oscillator.

c) Solve the resulting single differential equation for the motion of mass m_1 over a time interval of 20 units and plot the displacement of that mass, $x_1(t)$, over that time interval. Explain what you see.

d) In (13) use the values of $m_1 = 10$ and $k_1 = 90$ with $y(0) = 0$ and $y'(0) = 0$ and $f(t) = 10\cos(3t)$ and set $m_2 = 1$ with varying values of k_2. Keep $c_1 = 5$ and $c_2 = 0.5$. What do you observe in the maximum amplitude of the initial mass m_1 as one changes k_2? Defend your observation with data or plot.

e) From (d) what is the "best" value of k_2? Be sure you define the word "best."

f) For the best value of k_2 determine the maximum amplitude displacement for mass m_1, $x_1(t)$ over a range of frequencies.

Activity 3

We consider another, comparable configuration as Activity 2 for practice. Notice that mass m_2 is only 1% of mass m_1 which is quite realistice in structural design usingTMDs.

a) Determine the form of (13) when $c_1 = 5$ and $c_2 = 0.5$.

b) In the resulting system use $m_1 = 10$ and $k_1 = 90$ with $y(0) = 0$ and $y'(0) = 0$ while $f(t) = 10\cos(3t)$ and set $m_2 = 0.1$ with $k_2 = 0$, i.e. withdraw the second mass system. Explain the physical significance and impact results of the numbers 10 , 90, and 3 in the above sentence in terms of the motion of the initial oscillator.

c) Solve the resulting single differential equation for the motion of mass m_1 over a time interval of 20 units and plot the displacement of that mass, $x_1(t)$, over that time interval. Explain what you see.

d) In (13) use the values of $m_1 = 10$ and $k_1 = 90$ with $y(0) = 0$ and $y'(0) = 0$ and $f(t) = 5\cos(3t)$ and set $m_2 = 0.1$ with varying values of k_2. Keep $c_1 = 5$ and $c_2 = 0.5$. What do you observe in the maximum amplitude of the initial mass as one changes k_2? Defend your observation with data or plot.

e) From (d) what is the "best" value of k_2? Be sure you define the word "best."

f) For the best value of k_2 determine the maximum amplitude displacement for mass m_1, $x_1(t)$ over a range of frequencies.

Activity 4

From the introductory material for this scenario offered above and the analyses in Activities 2 and 3 offer a description of how to build a TMD to stop the resonance phenomena in the case of (14) where there is damping, i.e. $c_1 \neq 0$ and $c_2 \neq 0$,

$$
\begin{aligned}
m_1 x_1''(t) &= -k_1 x_1(t) - c_1 x_1'(t) - k_2(x_1(t) - x_2(t)) - c_2(x_1'(t) - x_2'(t)) + p_0 \cos(\omega t) \\
m_2 x_2''(t) &= -k_2(x_2(t) - x_1(t)) - c_2(x_2'(t) - x_1'(t))\,. \qquad (14)
\end{aligned}
$$

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