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SIMIODE Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations

STUDENT VERSION
CIRCUIT TUNER

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STATEMENT

Differential equations prove exceptional at modeling electrical circuits. In fact the very simple circuit, which is fundamental to larger circuit building, and three of the most fundamental electrical objects, a resistor, a capacitor, and inductor, can be modeled by a constant coefficient, linear, second order differential equation. Consider the circuit in Figure 1. The EMF $E(t)$ represents an Electromotive Force generated by an excess of electrons on one side of a barrier (the Switch) and a paucity of electrons on the other side of the barrier. When the switch is thrown the electrons in the excess area (say to the left of the circle marked EMF) seek to take the path of least resistance to get to the location of the paucity of electrons (to the right of the circle marked EMF). Thus we say there is a *potential* awaiting the switch to be thrown and that potential (just like water held high and then released to run down a descending track) causes electrons to flow clockwise through the capacitance (C), through the resistance (R), and finally through the inductance (L) until these electrons are "home" to the region of paucity of electrons.

Across each one of these devices (R, L, and C) there is a change in the potential or a voltage drop and the net change in potential around the circuit clockwise starting from the left of the EMF to the right of the EMF has to equal the potential across the EMF itself. It is as if water has a drop of h feet along the route from the left of EMF, through C, through R, and then through L, back to the right side of EMF. The total drop in height over this route is the sum of the three drops over C, R, and L. Thus h has to equal the sum of the drops over each of the elements, i.e. over C, over R, and over L. Calling these drops h_C, h_R, and h_L respectively, we then note that $h = h_C + h_R + h_L$.

The electric potential is measured in a unit called a *volt* and one can think of a volt as the amount of energy required to move a unit charge to a specific spot in a static electric field. Our learning curve here does not include carrying all the baggage of fully developing the notions surrounding electrical circuits, even of the elementary basics of the physical devices that make up a circuit.

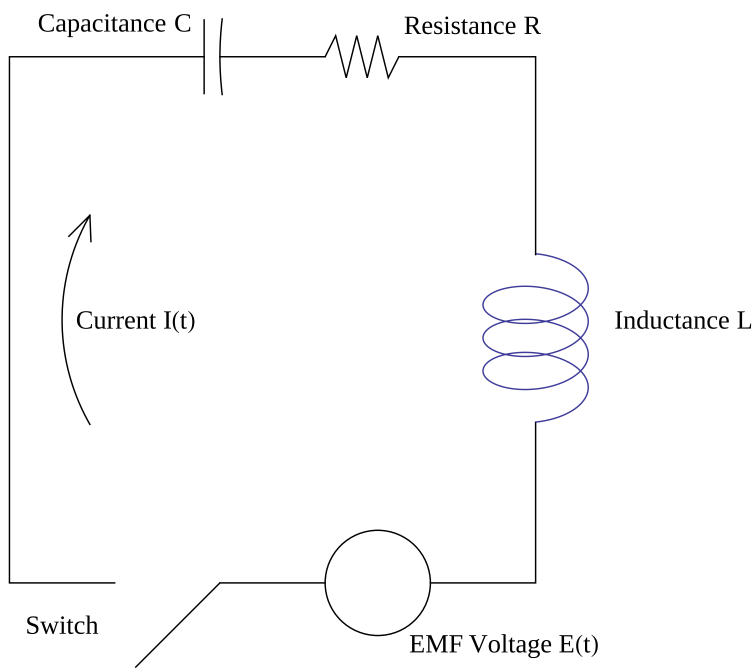

Figure 1. Diagram of an RLC circuit with imposed EMF $E(t)$ (generator or a battery) producing current, $I(t)$, in the circuit.

However, we will refer to some laws, first organized by Kirchhoff [1] and pertaining to the potential in a circuit.

Charge, $Q(t)$, is a measure of the number of electrons present and if we have 6.241×10^{18} electrons then we say we have one *coulomb* of charge. If charge passes by a point then we say we have a *current*. An *ampere* is a measure of the amount of electric charge (electrons) which passes by a point per unit time, or the current at a point in a circuit at time t. If some 6.241×10^{18} electrons pass a given point each second we will say there is a current flow of one *ampere*. Usually $I(t)$ refers to the amount of current (in amperes or amps) passing by the point or in that branch of the circuit at time t.

First we need to do some identification of the potential or voltage drop across each of the three devises we are using: capacitance (C), resistance (R), and inductance (L). Each of these devices has a unique physical construction, each is rated with different units, each deals with the electrons coming into it differently and the effect the device has on these electrons. Each has a potential or voltage drop across it and Kirchhoff's Voltage Law [1] says that the sum of the voltages in a circuit is equal to that of the induced $\text{Emf}(t)$ voltage at any time t in seconds. To get this sum we need the voltage drop across each device.

(C) The voltage (in volts) drop, E_C, across a capacitor rated at C *farads* is

$$E_C = \frac{1}{C} \cdot Q(t)\,. \tag{1}$$

(R) The voltage (in volts) drop, E_R,across a resistor rated at R *ohms* is

$$E_R = R \cdot \frac{dQ(t)}{dt} = R \cdot I(t)\,. \tag{2}$$

(L) The voltage (in henry) drop, E_L, across an inductor rated at L *henrys* is

$$E_L = L \cdot \frac{dI(t)}{dt}\,. \tag{3}$$

Thus from expressions (1), (2), and (3) and the fact that these voltages have to sum up to equal the induced voltage, $\mathrm{Emf}(t)$, across the circuit, we have a differential equation (4) describing charge, $Q(t)$, in a circuit depicted in Figure 1, for we note $I(t) = \frac{dQ(t)}{dt}$ and hence $\frac{dI(t)}{dt} = \frac{dQ^2(t)}{dt^2}$.

$$E_C + E_R + E_L = \frac{1}{C} \cdot Q(t) + R \cdot \frac{dQ(t)}{dt} + L \cdot \frac{dI(t)}{dt} = \mathrm{Emf}(t)\,. \tag{4}$$

Now watch the magic as we differentiate both sides of (4) to obtain something akin to a familiar looking differential equation (4).

$$\frac{1}{C} \cdot \frac{dQ(t)}{dt} + R \cdot \frac{dQ^2(t)}{dt^2} + L \cdot \frac{dI^2(t)}{dt^2} = \mathrm{Emf}'(t)\,. \tag{5}$$

Finally, writing the variables in terms of $I(t)$ and not $I(t)$ *and* $Q(t)$ we see (6)

$$L \cdot \frac{dI^2(t)}{dt^2} + R \cdot \frac{dI(t)}{dt} + \frac{1}{C} \cdot I(t) = \mathrm{Emf}'(t)\,. \tag{6}$$

Of course we will have to tell the circuit what initial current is in it, i.e. $I(0) = 0$ usually until we turn the switch on and also $I'(t) = 0$. There you have it, differential equation (6) is exactly the same as differential equation (7) for the spring mass dashpot system with appropriate terms identified. Can you identify the right terms' correspondences?

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = f(t)\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{7}$$

If we make the identifications in Table 1 all the tools we develop for the mechanical motion situation of spring mass dashpots will be useful in our study of RLC circuits and vice-versa.

Of course, this RLC circuit equation will need a non-homogeneous term, $\mathrm{Emf}(t)$ to supply electrons and we will need to address the solution of such equations. Let us look at one circuit before leaving them sit for a while.

Spring Mass Dashpot	RLC Circuit
mass m	inductance L
damping or resistance c	resistance R
spring constant k	inverse of capacitance $\frac{1}{C}$
forcing function $f(t)$	derivative of induced voltage $\text{Emf}'(t)$

Table 1. Comparison of terms between Spring Mass Dashpot and RLC Circuit differential equation.

Simple RLC Circuit Model, Solution, and Interpretation

We now examine a circuit in which a current is present and does not have a driving $\text{Emf}(t)$, expecting things to dampen out, in this case current to run out.

Again, let us consider an RLC circuit as depicted in Figure 2 in which we have an initial current, $I(0) = 3.2$ amps with a resistance of $R = 7$ ohms, an inductance of $L = 1$ henry, and a capacitance of $C = 0.1$ farads. Since we have some current in the circuit already $I(0) = 3.2 > 0$ at the start we shall not need an inducing $\text{Emf}(t)$, so $\text{Emf}(t) = 0$. Let us see what happens to the current in the circuit by solving the appropriate RLC circuit differential equation (8)

$$\frac{1}{0.1} \cdot I(t) + 7 \cdot \frac{dI(t)}{dt} + 1 \cdot \frac{dI^2(t)}{dt^2} = \text{Emf}'(t) = 0\,, \quad I(0) = 3.2 \quad \text{and} \quad I'(0) = 0\,. \tag{8}$$

or as we are accustomed to reading it

$$1 \cdot \frac{dI^2(t)}{dt^2} + 7 \cdot \frac{dI(t)}{dt} + 10 \cdot I(t) = 0\,, \quad I(0) = 37 \quad \text{and} \quad I'(0) = 0\,. \tag{9}$$

Activity 0

a) Solve the RLC circuit differential equation (9) for $I(t)$. Warning for *Mathematica* users: *Mathematica* reserves the variable `I` as the symbol for i, the imaginary square root of -1, so here we will use $x(t) = I(t)$ for the current in the circuit.

b) Consider the values of R to be 0.007, 0.07, 0.7, 7, 70, and then 700, and solve (9) in each case, keeping all other values the same, and plot the solution for the current in the circuit over the time interval $[0, 25]$ s with a vertical plot interval $[-3, 3]$ in each case. Identify each plot with its associated R value and describe what is happening to the current, $I(t)$, in each corresponding circuit over time, t.

Moving On

Let us consider the differential equation (10) for an RLC circuit (see Figure 2) with driving voltage $\text{Emf}(t) = \sin(wt)$ (current $I(t) = y(t)$):

$$L \cdot y''(t) + R \cdot y'(t) + \frac{1}{C} \cdot y(t) = \text{Emf}'(t) = \omega \cos(\omega t) \tag{10}$$

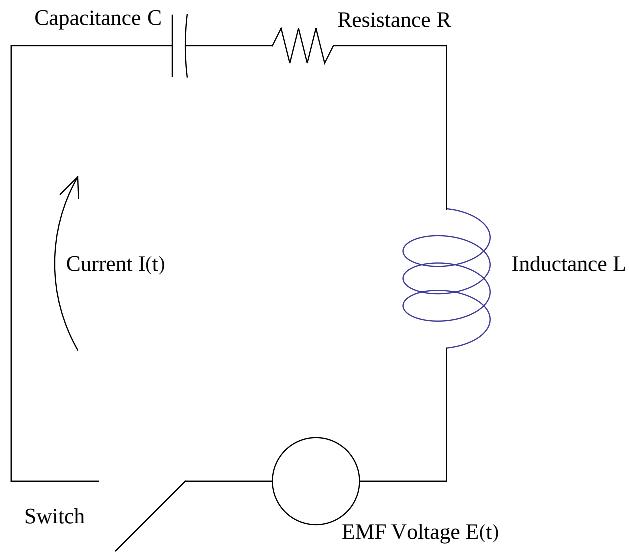

Figure 2. Diagram of an RLC circuit with imposed EMF $E(t)$ (generator or a battery) producing current, $I(t)$, in the circuit.

As an example of the power of computer algebra systems, here is the *Mathematica* code for solving and grabbing the solution of (10):

```
ys[t_,R_,L_,C_,ω_] = y[t]/.DSolve[{L y''[t]+R y'[t]+1/C y[t]
            == D[Emf[t, ω], t], y[0] == 0, y'[0] == 0}, y[t], t][[1]],
```

with output about a half page long! We do not repeat it here though. However, in this solution there is the transient solution involving R, L, and C with no ω's and then there is the steady state solution (11):

$$y_{\text{steady state}}(t) = \frac{C\omega\left(\left(1 - CL\omega^2\right)\cos(\omega t) + CR\omega\sin(\omega t)\right)}{C^2\omega^2\left(L^2\omega^2 + R^2\right) - 2CL\omega^2 + 1} \tag{11}$$

We would like to study what is called *gain*. Gain is the ratio of the amplitude of the steady state output voltage, V_{out}, to the amplitude of th input voltage , V_{in}, i.e. gain $= \frac{V_{\text{out}}}{V_{\text{in}}}$. We measure V_{out} as the amplitude of the steady state voltage across the resistance R in the circuit, i.e. $V_{\text{out}} = R \cdot \text{Amplitude}(y_{\text{steady state}}(t))$. In our case V_{in} is just 1, for $\text{Emf}(t) = \sin(\omega t)$.

Trigonometry Pause and Phase Angle

In our solutions for a second order (12),

$$a \cdot y''(t) + b \cdot y'(t) + c \cdot y(t) = 0\,, \quad y(0) = y_0 \text{ and } y'(0) = 0\,, \tag{12}$$

the case where the discriminant, $b^2 - 4 \cdot a \cdot c$ is negative gives rise to complex roots to the characteristic equation and the general solution looks like (13)

$$\begin{aligned} y(t) &= Ae^{\left(\frac{-b}{2a}t\right)} \sin\left(\frac{\sqrt{4ac-b^2}}{2a}t\right) + Be^{\left(\frac{-b}{2a}t\right)} \cos\left(\frac{\sqrt{4ac-b^2}}{2a}t\right) \\ &= e^{\left(\frac{-b}{2a}t\right)} \left(A\sin\left(\frac{\sqrt{4ac-b^2}}{2a}t\right) + B\cos\left(\frac{\sqrt{4ac-b^2}}{2a}t\right)\right) \end{aligned} \tag{13}$$

If we let $\left(\frac{\sqrt{4ac-b^2}}{2a}\right) = \omega$ then (13) simplifies to (14)

$$y(t) = e^{\left(\frac{-b}{2a}t\right)} \left(A\sin(\omega t) + B\cos(\omega t)\right) . \tag{14}$$

We wish to combine the sine and cosine terms in (14) into one sine function with a phase angle. We can do this because of the trigonometric identity:

$$\sin(x+y) = \sin(x)\cdot\cos(y) + \cos(x)\cdot\sin(y) ,$$

and the diagram in Figure 3. Identifying $x = \omega t$ and $y = \theta$ and multiplying and dividing each expression by $\sqrt{A^2+B^2}$ we can convert the expression $A\sin(\omega t) + B\cos(\omega t)$ to

$$\begin{aligned} &\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin(\omega t) + \frac{B}{\sqrt{A^2+B^2}}\cos(\omega t)\right) \\ &= \sqrt{A^2+B^2}\left(\cos(\theta)\sin(\omega t) + \sin(\theta)\cos(\omega t)\right) \\ &= \sqrt{A^2+B^2}\left(\sin(\omega t + \theta)\right) \end{aligned} \tag{15}$$

where $\theta = \text{Arctan}\left(\frac{B}{A}\right)$ is called the *phase shift*.

From (15) we can write our solution (14) as (16),

$$y(t) = e^{\left(\frac{-b}{2a}t\right)} \sqrt{A^2+B^2}\left(\sin(\omega t + \theta)\right) . \tag{16}$$

The phase angle, θ permits us to see the solution as a $\sin(\omega t)$ type function but out of phase by θ radians. More precisely, we can write

$$\sin(\omega t + \theta) = \sin(\omega\left(t + \frac{\theta}{\omega}\right)) , \tag{17}$$

and refer to $\frac{\theta}{\omega}$ as the *phase angle*.

Incidentally, this combining of terms permits us also to see the amplitude of $y(t)$ in (16) as $\sqrt{A^2+B^2}$ which decays in view of the term $e^{\left(\frac{-b}{2a}t\right)}$.

Consider the differential equation (18)

$$1 \cdot y''(t) + 6 \cdot y'(t) + 25 \cdot y(t) = 0 \quad y(0) = 1 \text{ and } y'(0) = 0 \tag{18}$$

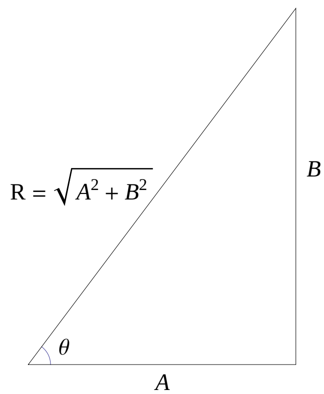

Figure 3. Useful triangle diagram in rearranging an oscillating solution to a second order differential equation which results in a single phase shifted solution from the sum of sine and cosine terms in solution.

whose solution is

$$\mathrm{ysol}(t) = \frac{1}{4}e^{-3t}(3\sin(4t) + 4\cos(4t))\,.$$

while the phase shifted form of the solution (16) is

$$\begin{aligned}\mathrm{ysol}(t) &= \frac{5}{4}e^{-3t}\sin\left(4t + \tan^{-1}\left(\frac{4}{3}\right)\right)\\ &= \frac{5}{4}e^{-3t}\sin\left(4\left(t + \frac{\tan^{-1}\left(\frac{4}{3}\right)}{4}\right)\right)\\ &= \frac{5}{4}e^{-3t}\sin\left(4\left(t + 0.231824\right)\right)\,. \end{aligned} \tag{19}$$

In (19) we note that $\omega = 4$ and $\theta = \arctan\left(\frac{4}{3}\right) = 0.231824$ radians.

We see in Figure 4 that the phase angle is depicted by the shift between the two functions.

Activity 1

a) Solve the differential equation (20),

$$1 \cdot y''(t) + 10 \cdot y'(t) + 29 \cdot y(t) = 0\,, \quad y(0) = 1 \text{ and } y'(0) = 0\,. \tag{20}$$

b) Convert the solution to phase angle format and compute the phase angle θ in radians.

c) Plot both solutions in (a) and (b) on the same axis over the interval $[0, 2]$ to confirm your analyses. What should you see?

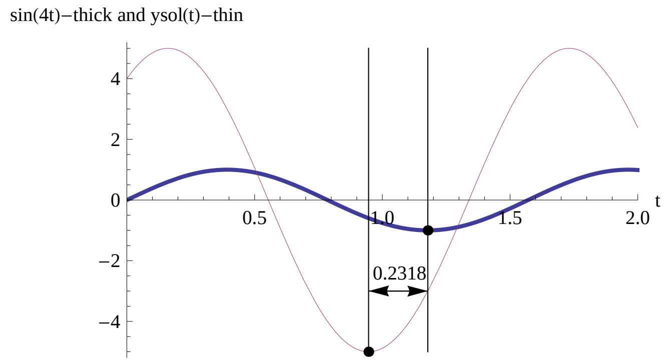

Figure 4. A plot of the oscillating portion (not damped), $3\sin(4t) + 4\cos(4t)$, of the solution (thin) to a second order differential equation (16) with its simple frequency curve $\sin(\omega t)$ (thick). Notice the phase angle here of 0.2318 radians from bottom to bottom illustrating what we mean by out of phase by a phase angle of 0.2318 radians.

Back to the Circuit

In our study of phase angle representation in the previous section we saw that the $\sin(wt)$ and $\cos(wt)$ terms of (11) could be combined into one sine term (albeit with a phase angle) with one amplitude. This amplitude, shown in (21), we call $\text{Amplitude}(y_{\text{steady state}}(t))$

$$\text{Amplitude}(y_{\text{steady state}}(t)) = R\sqrt{\frac{C^2\omega^2}{C^2\omega^2\left(L^2\omega^2 + R^2\right) - 2CL\omega^2 + 1}}\,. \tag{21}$$

Thus our gain (recall, gain is the ratio of the amplitude of the steady state output voltage , V_{out}, to the amplitude of the input voltage, V_{in}) is given by (22). Here, $V_{\text{in}} = 1$ for $\text{Emf}(t) = \sin(\omega t)$ and has amplitude 1.

$$\text{gain} = R\sqrt{\frac{C^2\omega^2}{C^2\omega^2\left(L^2\omega^2 + R^2\right) - 2CL\omega^2 + 1}}\,. \tag{22}$$

Notice that our gain is a function of R, L, C, and ω. This gain is a measure of the response of the circuit to input voltage $\text{Emf}(t)$ in this case $\text{Emf}(t) = \sin(\omega t)$.

Let us fix R at 1 ohm and L at 1 henry and see what gain is in this case as a function of C over a range of ω values. Let us "tune" this circuit by changing C, the size of the capacitance in the circuit and see how gain changes as input voltage frequency, ω, changes.

Figure 5 illustrates the power of differential equation modeling. For we can alter parameters in our equation and see the results in a physical (in this case electrical) system. Indeed, we see in this plot that for a capacitance of $C = 0.0005$ farads if we have an input voltage with a frequency around $\omega = 45$ (44.7214 to be precise) then the gain is greatest. Optimization is a calculus problem and we merely have to take the derivative of gain with respect to ω and find where it is 0. All other frequencies give smaller gain for this particular capacitance. In fact, we can say that as we decrease

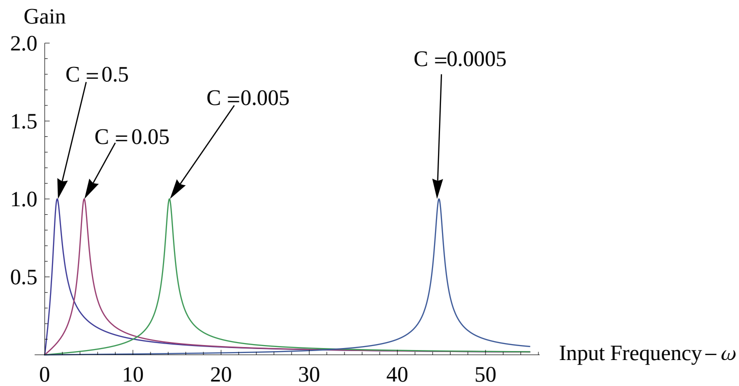

Figure 5. Plot of gain as a function of input frequency, ω, for various values of capacitance, $C = 0.5, 0.05, 0.005, 0.0005$ farads in (10) with R at 1 ohm and L at 1 henry.

our capacitance the optimal frequency, i.e. frequency which gives highest gain, decreases and we might want to look into this for a more exact relationship. We shall do that in Activity 2 below.

Put another way, we see that if our input voltage has a specific frequency, ω, there is a unique capacitance, C, for this circuit that will maximize our gain. By changing C we can tune our circuit to maximize gain for a given input frequency, ω. This is, in fact, how we tune a radio, for we change the capacitance of the radio's circuit so as to maximize the gain for the frequency (on our dial) that we wish to hear. So, the next time you try to find the station where Cousin Brucie is dedicating a Top Ten song from "Billie Bob" to "Sally May" know that a differential equation describes exactly what you are doing. How's that for cool?

Activity 2

Use your understanding of RLC circuits to show for an imposed $\mathrm{Emf}(t) = \sin(\omega t)$ on the RLC circuit given by (23) the maximum gain is obtained when $\omega = \frac{1}{\sqrt{LC}}$ and thus we could tune our radio by changing the inductance L as well, If that were as convenient as changing the capacitance, which it is not. So let us stick to tuning by changing the capacitance C.

$$L \cdot y''(t) + R \cdot y'(t) + \frac{1}{C} \cdot y(t) = \mathrm{Emf}'(t) = \omega \cos(\omega t)\,, \quad y'(0) = 0 \quad y(0) = 0 \tag{23}$$

Activity 3 - Tune the Radio

The Amplitude Modulated (AM) radio carrier frequencies are in the frequency range 535-1605 kHz. 1 Hz means 1 cycle per second while kHz means 1,000 cycles per second. The unit Hz is called a *hertz*. Carrier frequencies of 540 to 1600 kHz are assigned at 10 kHz intervals. The (Frequency Modulated (FM) radio band is from 88 to 108 MHz [2]. Recall 1 kHz means 1,000 cycles per second. So 660 kHz is oscillation at the rate of 660,000 cycles per second. We offer up a "radio" (24) and ask you to tune in several stations by changing the capacitance and computing the optimal gain for these stations. Recall gain is the ratio of the amplitude of the steady state output voltage , V_{out}, to the amplitude of the input voltage, V_{in}, i.e. gain $= \frac{V_{\text{out}}}{V_{\text{in}}}$. We measure V_{out} as the amplitude of the steady state voltage across the resistance R in the circuit, i.e. $V_{\text{out}} = R \cdot \text{Amplitude}(y_{\text{steady state}}(t))$. In our case V_{in} is just 1, for $\text{Emf}(t) = \sin(\omega t)$.

So, we consider our radio circuit (24):

$$L \cdot \frac{dI^2(t)}{dt^2} + R \cdot \frac{dI(t)}{dt} + \frac{1}{C} \cdot I(t) = \text{Emf}'(t)\,. \tag{24}$$

We will have to tell the circuit what initial current is present, i.e. $I(0) = 0$ usually until we turn the switch on and also we can presume ther is no change in the current intially, i.e. $I'(t) = 0$. Let us build this radio with the following values $L = 0.033$ henrys, $R = 100$ ohms, and C in farads can vary as needed to tune to various input frequencies ω. We note that if we wish to have, say 540 kHz then $\omega = 540,000 \cdot 2\pi$, and in general to have x kHz we will need $\omega = x \cdot 1000 \cdot 2\pi$.

a) Solve the differential equation (24) for the radio circuit.

b) Collect the coefficients A and B of the steady state $\sin(\omega t)$ and $\cos(\omega t)$ terms, respectively. Show that all other terms will dissipate, i.e. go to zero quickly, leaving only $\sin(\omega t)$ and $\cos(\omega t)$ terms.

c) Using the information above compute the gain as a function of capacitance C and input voltage frequency ω.

d) For a given input voltage frequency ω determine the maximum gain for this circuit.

e) For several AM frequencies, say 540 kHz ($\omega = 540000 \cdot 2 \cdot \pi$), 880 kHz ($\omega = 880000 \cdot 2 \cdot \pi$), and 1520 kHz ($\omega = 1520000 \cdot 2 \cdot \pi$), plot gain as a function of the capacitance C to demonstrate that your maximum gain is what your formula in (c) predicts and that your radio is tuned in.

REFERENCES

[1] Kirchhoff's circuit laws. `https://en.wikipedia.org/w/index.php?title=Kirchhoff's_circuit_laws\&oldid=737191986`. Accessed 9 September 2016.

[2] HyperPhysics. 2009. AM and FM Radio Frequencies. `http://hyperphysics.phy-astr.gsu.edu/hbase/audio/radio.html` Accessed 18 September 2016.