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# STUDENT VERSION
# Container Shape Falling Water

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## STATEMENT

### Introduction

We consider the question. For a fixed volume of water, say $V = 1000\text{cm}^3$ and fixed exit hole at the bottom of a container does the shape of the column matter in determining the time of complete draining of the water from the container?

We examine many different containers, each holding 1000 $\text{cm}^3$ of water and each with an exit hole for the water to fall out of the container of radius $r = 0.25$ cm. If $\alpha$ is the effective area (percentage due to friction at the opening called the *discharge or contraction coefficient*) for water flowing through the exit hole then $\alpha = 0.70$.

First we need to recall the governing equations or model for the height of a column of liquid in a cone with a hole in the bottom from which the liquid leaks out.

Per this analysis in [1], we calculate the loss of water out of the container in two different ways and set them equal using Torricelli' s Law development as outlined in [1].

Here are the definitions and specifications for all the containers we shall work with:

- $h(t)$ is the height of the column of water in the container at time $t$. Length units will be in centimeters and time units in seconds.
- $A(h(t))$ is the cross sectional are of our container at height $h(t)$.
- $r_c = 0.25$ cm is the radius of the small bore hole through which the water exits our container.
- $\alpha = 0.70$ is the effective area (percentage of water that actually gets through the exit hole due to friction at the opening called the *discharge or contraction coefficient*.

(L) At time $t$ a small element of height changes, $h'(t)$, over a cross sectional area $A(h(t))$. Hence we will have lost

$$A(h(t)) \cdot h'(t) \tag{1}$$

volume of water.

(R) Since the water is flowing out of the opening with a velocity (according to Torricelli's Law) of $\sqrt{2\,g\,h(t)}$ m/s and our opening's cross sectional area is $a = \pi r_c^2$ m$^2$ then we are losing water at a rate of

$$\alpha\, a\,\sqrt{2\,g\,h(t)} = \alpha\,\pi(r_c)^2\sqrt{2\,g\,h(t)}\,\mathrm{m}^3/\mathrm{s}\,, \tag{2}$$

where we recall the effective area for fluid flow through the opening is only a fraction ($\alpha = .7$) of the area.

Now if we equate the computations for the loss of water found in (1) with that computed in (2) we obtain the following differential equation, with our initial height $h(0) = h_0$ m as an initial condition:

$$A(h(t)) \cdot h'(t) = \alpha\,\pi(r_c)^2\sqrt{2\,g\,h(t)}\,, \quad h(0) = h_0\,. \tag{3}$$

Equation (3) is one form of Torricelli's Law for the height of falling column of water.

**Activity 1**

a) Consider a right circular cylinder of a fixed volume $V = 1000\,\mathrm{cm}^3$ of water in it and exit hole of radius$r = 0.25$ cm, with discharge or contraction coefficient $\alpha = 0.70$. Determine the time it takes to empty this container.

b) Now if we presume the volume of water in our cone is $V = 1000\,\mathrm{cm}^3$. Then we have a ratio between height and radius at every given height, for the volume of a cone of height $H$ and radius $R$ is $V = \pi R^2 H$. This leads to a relationship between $R$ and $H$ when the volume is fixed at $V = 1000\,\mathrm{cm}^3$. For a range of heights, $0.01 \le H \le 100$ determine the time it takes to empty the tank and offer a plot of the time it takes to empty the tank for a given value of $H$ vs. $H$ and comment on the shape of the plot.

**Activity 2**

Consider a right circular cylinder of radius $R$ cm and height $H$ cm which contains $1000\,\mathrm{cm}^3$. Note that $R$ and $H$ are related through the formula for the volume of a right circular cylinder of radius $R$ and height $H$, i.e. $V = \pi R^2 H$. As such determine the time it takes to empty the container of water over a range of values of $H$, say $.01 \le H \le 100$. How do you think the time to empty the container changes (or does it?) as $H$ increases? Plot time to empty the container as a function of height $H$. Comment on your plot and what it says physically.

### Activity 3

Pick the dimensions of a right circular cone of radius $R$ and height $H$ that points downward which contains $1000\,\mathrm{cm}^3$ and use the specifications outlined above. For $H = 20$ cm determine the time it takes to empty your container of water.

### Activity 4

a) Pick the dimensions of a right circular cone of radius $R$ and height $H$ that points upward which contains $1000\,\mathrm{cm}^3$ and use the specifications outlined above. Determine the time it takes to empty your container of water.

b) Note that $R$ and $H$ are related through the formula for the volume of a right circular cone of radius $R$ and height $H$, i.e. $V = \frac{1}{3}\pi R^2 H$. As such determine the time it takes to empty the container of water over a range of values of $H$, say $.01 \leq H \leq 100$. How do you think the time to empty the container changes (or does it?) as $H$ increases? Plot time to empty the container as a function of height $H$ and comment on your plot.

### Activity 5

Consider a container in the shape of a sphere which has volume $V = 1000\,\mathrm{cm}^3$. Determine the time it takes to empty the tank if initially filled with water.

### Activity 6

a) Pick the dimensions of a right circular cone of radius $R$ and height $H$ that points downward which contains $1000\,\mathrm{cm}^3$ and use the specifications outlined above. Determine the time it takes to empty your container of water.

b) Note that $R$ and $H$ are related through the formula for the volume of a right circular cone of radius $R$ and height $H$, i.e. $V = \frac{1}{3}\pi R^2 H$. As such determine the time it takes to empty the container of water over a range of values of $H$, say $.01 \leq H \leq 100$. How do you think the time to empty the container changes (or does it?) as $H$ increases? Plot time to empty the container as a function of height $H$ and comment on your plot.

### Activity 7

Create a question of your own., and render an analysis? E.g., other shapes, comparing emptying times for different shapes, for a given shape vary the exit hole radius.

### REFERENCES

[1] Winkel, Brian. 2015. 1-15-T-Torricelli. `https://www.simiode.org/resources/488`. Accessed 21 November 2016.