www.simiode.org
SIMIODE
A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
BUILDING WATER CLOCKS

Sania Qureshi
Basic Sciences and Related Studies
Mehran University of Engineering and Technology
Jamshoro, Sindh PAKISTAN

Brian Winkel
SIMIODE
26 Broadway
Cornwall NY USA

STATEMENT

Modeling Activities

1. Suppose we are given a *right circular cylinder* with spigot hole for water to exit at the bottom. What shape container would we have to build so that the water falls at a constant rate, meaning for a right circular cylinder what cross-sectional area $A(h)$ at height h would we have to build so that the height of the water falls at a constant rate?
2. We consider the problem of determining the area of the opening or spigot hole, $s(t)$, at the bottom of a *right circular cylinder* of water of constant cross sectional area $A(h(t)) = K$ so that the water falls at a constant rate, i.e. $h'(t) = c$ $(c < 0)$, where $h(t)$ is the height of the water at time t and $A(h(t))$ is the cross sectional area of the cylinder at height $h(t)$.
3. We seek to determine the area of the opening or spigot hole, $s(t)$, at the bottom of an *inverted right circular cone* of water of where the cone has a base radius of R and height of H so that the water falls at a constant rate, i.e. $h'(t) = c$ $(c < 0)$, where $h(t)$ is the height of the water at time t and $A(h(t))$ is the cross sectional area of the cone at height $h(t)$.
4. We consider the problem of determining the area of the opening or spigot hole, $s(t)$, of the spigot at the bottom of an *inverted hemisphere* of water of where the hemisphere has a radius of R and height of H so that the water falls at a constant rate, i.e. $h'(t) = c$ $(c < 0)$, where $h(t)$ is the height of the water at time t and $A(h(t))$ is the cross sectional area of the hemisphere at height $h(t)$.

Modeling Activity (1) you will find to be rather straightforward and offer up a nice solution. However, we have not made any progress on Modeling Activities (2) - (4) and we believe these to be inverse problems which can be hard. Nevertheless, we pose them for completeness and hope that students might make progress, perhaps even numerically. Indeed, one can ask information about the shape of the cylinder and the changing spigot aperture for any rate of fall in the height to be most general.

APPENDIX

Derivation of Torricelli's Law from First Principles

We reproduce here the derivation for the rate at which water flows out of a tank of water through a spigot hole in the bottom. This is known as Torricelli's Law and a derivation is found in [1]. We examine a model based on first principles, not just on empirical fit. Given a column of water in a cylinder with a tiny hole at the bottom of the cylinder through which the water exits, how long does it take to empty the cylinder down to the level of the hole? More specifically, if we are given the cross sectional area of the cylinder as a function of height and the area of the tiny hole at the bottom of the cylinder can we model the outflow of the water from the cylinder.

Now for some basic physics background. We first consider a law of physics which can help us. The Law of Conservation of Energy says that the sum of the potential energy and the kinetic energy of a particle of mass m is constant, i.e., is conserved. So if we consider a particle of water of mass m initially atop a cylinder of water, some h meters above a small, sharp-edged opening in the side wall of the cylinder through which the water can exit the cylinder, this mass of water has initial potential energy $m * g * h$ (PE_i), where g is the acceleration due to gravity, and initial kinetic energy $\frac{1}{2}mv_i^2$ (KE_i), where v_i is the velocity of the mass initially. This gives us a total energy of TE_i, initially, when the mass of water is on the top of the cylinder of water:

$$TE_i = KE_i + PE_i = \frac{1}{2}mv_i^2 + mgh\,.$$

When this mass of water reaches the opening it has height 0 meters above the opening and a final velocity of v_f. Hence, the total energy at the final time (TE_f) the mass reaches the opening is

$$TE_f = KE_f + PE_f = \frac{1}{2}mv_f^2 + mg * 0 = \frac{1}{2}mv_f^2\,.$$

Now, by the Law Conservation of Energy we know $TE_i = TE_f$ and thus we have

$$\frac{1}{2}mv_i^2 + mgh = \frac{1}{2}mv_f^2\,,$$

which when one solves for v_f yields $v_f = \sqrt{2gh + v_i^2}$. In particular, if our element of mass of water was at rest initially, i.e. $v_i = 0$ we have the classical Torricelli's Law

$$v_f = \sqrt{2gh}\,,$$

where v_f is the speed of the element of mass of water as it exits the opening in the cylinder. In particular we have the following differential equation for $v_f = h'(t)$:

$$h'(t) = \sqrt{2gh(t)}\,, \tag{1}$$

where $h(t)$ is the height of the element of mass of water.

You might solve this differential equation (1) if we know the height of the column of water, initially, i.e., $h(0) = h_0$ and see whether or not its form makes sense.

Modeling the outflow from a cylinder of water

We offer some activities using Torricelli's Law as a practice for the activities of this modeling scenario.

We note that (6) below would be a typical differential equation for the height of the water in our container given the geometry of the situation described leading up to (6). For the activities in this modeling scenario the geometry and hence the terms of the appropriate differential equation similar to (6) would be different.

Suppose we have a cylinder of height 0.20 m and radius 0.05 m and it is filled with water. There is an opening in the side of the cylinder at the bottom of the cylinder in the form of a sharp edged circle of radius $r = 0.01$ m.

Empirically, it is known that while the area of the opening is $\pi r^2 = \pi(.01)^2$ m^2 the effective area for fluid flow through the opening is only a fraction of the area. That fraction, α, is called the *discharge or contraction coefficient*. Let us assume $\alpha = 0.7$, which is a reasonable value from the literature for various opening shapes and sizes.

We now build a differential equation to determine the height of the water in the cylinder as a function of time, $h(t)$ in meters – in this case. To do this we need to use accounting!! This will prove to be a nice way to build models, namely compute some quantity in two different ways and set the two different computations equal to each other. If the computations involve derivatives we have built a differential equation!

So let us compute, in two different ways, the rate at which the volume of water changes at time t. We shall do this for a general cross sectional area of our cylinder as a function of height, $A(h(t))$.

(L) At time t a small element of height changed, $h'(t)$, over a cross sectional area $A(h(t))$. Hence we will have lost

$$A(h(t)) * h'(t) \tag{2}$$

volume of water. In our case, for our circular cylinder of constant radius 0.05 m, we have the following computation:

$$A(h(t)) * h'(t) = \pi(.05)^2 h'(t) \tag{3}$$

(R) Since the water is flowing out of the opening with a velocity (according to Torricelli's Law) of $\sqrt{2gh(t)}$ m/s and our opening's cross sectional area is $a = \pi r^2 = \pi(.01)^2$ m^2 then we are losing water at a rate of

$$\alpha a\sqrt{2gh(t)} = \alpha\pi r^2\sqrt{2gh(t)} = .7\pi(.01)^2\sqrt{2gh(t)} \quad \mathrm{m}^3/\mathrm{s}\,, \tag{4}$$

where we recall the effective area for fluid flow through the opening is only a fraction ($\alpha = .7$) of the area.

Now if we equate the computations for the loss of water found in (2) with that computed in (4) we obtain the following differential equation (adding our initial height $h(0) = h_0 = 0.20$ m as an initial condition):

$$A(h(t)) * h'(t) = \alpha\, a\sqrt{2gh(t)}\,, \quad h(0) = h_0 \tag{5}$$

in general, and in our case, using $g = 9.8$ m/s^2,

$$\pi(.05)^2 h'(t) = .7\pi(.01)^2\sqrt{2(9.8)h(t)}\,, \quad h(0) = 0.20\,. \tag{6}$$

We repeat the activities from [1] to give some idea of what can be accomplished with the resulting models.

(1) Solve the differential equation (6) (be sure you pick the correct branch of the solution) and determine how long it takes to empty the cylinder of water. Plot the height $h(t)$ as a function of time, t, for the duration of the draining.

(2) Suppose $\alpha = 0.6$ instead of 0.7, then how long would it take to empty the cylinder of water? Plot the height $h(t)$ as a function of time, t, for the duration of the draining. Compare this answer and the plot with those from (1). Explain why this is reasonable.

(3) Continuing with (2), determine the velocity of the surface of the water and plot it over the duration of the draining. Explain why what you see is reasonable.

REFERENCES

[1] Winkel, B. 2015. 1-15-S-Torricelli. `https://www.simiode.org/resources/488`. Accessed 13 August 2016.