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STUDENT VERSION
Bottle Water Flow

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STATEMENT

Confession

There is a wonderful publication called *The Journal of Irreproducible Results*[1]. Here is what the self-effacing editors say about the journal:

> In six funny issues a year, *JIR* offers spoofs, parodies, whimsies, burlesques, lampoons, and satires. *JIR* appeals to scientists, doctors, science teachers, and word-lovers. *JIR* targets hypocrisy, arrogance, and ostentatious sesquipedalian circumlocution. We're a friendly escape from the harsh and the hassle. *JIR* makes you feel good.

Many years ago there was a piece in *JIR* which came out of a lab where an "infinite number of monkeys" were typing and one of them produced a book review about an absolutely wonderful book. The only problem was that the book had not been written yet! So in the style of *JIR* a call for the author was put out to write the book. Well I thought it was funny! You can check out their website `www.jir.com` to see many examples of very clever humor and spoofing.

In my 50 plus years of teaching I have acquired a great many articles and in looking through my files recently I came upon a three page typewritten document with the title, "MATH 360 Project #1 – Torricelli's Law for Water Flow." So we publish it here for you to learn differential equations using a model and your own data. We reproduce the project exactly as it was offered and we hope in the spirit of *JIR* that the true author will emerge.

Project #1 – Torricelli's Law for Water Flow

You are to perform an experiment and then formulate a model to explain the data. In deriving the model, you will be using a formula which has an unknown (to you) constant coefficient. One of the goals of the project is to use your computed mathematical model, and your experimental data, to find an estimate for this constant.

I. The Experiment

Take a transparent plastic bottle (e.g., a soda bottle) which has vertical sides of at least 10 cm, and which has a circular horizontal cross section. Carefully measure the circumference of the part with vertical sides. Make a hole about 2 mm in diameter in the side of the bottle near the bottom of the part with vertical sides. Carefully measure the diameter of the hole (or of the needle, ice pick, or whatever instrument that made it). At least give a close estimate (e.g., "between 1.8 and 2 mm"). The shape of the cross section, and your measurements, will only be needed at the very end of the project.

Fix a ruler, scaled in cm, to the side of the bottle, with 0 marking the height of the hole:

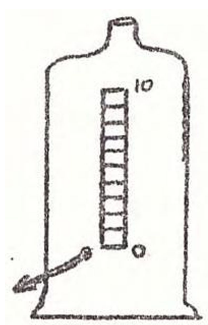

Figure 1. Sketch of apparatus.

Fill the bottle with water (leave it open on top), and, as the water runs out the hole, note the time (in secs) at which the water height reaches each cm, from 9 through 1. Let $t = 0$ be where the height is 10 cm.

Here is an example of data collected:

Height (cm)	10	9	8	7	6	5	4	3	2	1
Time (sec)	0	12	25	37	51	67	84	104	130	163

Your figures will not be the same as these unless you are using an identical bottle and the same size hole.

Also note the height and time at which the flow ceases to be a jet and starts to just dribble down the side. After a little practice, repeat the experiment three or four times and take the average (mean) of the readings at each height. Submit a table showing each of your readings, as well as the averages

II. The Model

We seek a mathematical model which will explain how the water height is a function of time. The important variables are

t, time since water started to flow;

$h(t)$, height of water in bottle above outlet hole;

$u(t)$, volume of water in bottle; and

$v(t)$, volume rate of flow through the outlet hole.

Three important constants are

u_0, volume of water in the bottle beneath the outlet hole;

A, horizontal cross-sectional area of the part with vertical sides;

k, area of the outlet hole.

1) Explain why $\frac{du}{dt} = -v(t)$.
2) Find $u(t)$ in terms of $h(t)$, A, and u_0, and show that

$$A\frac{dh}{dt} = -v(t)\,. \tag{1}$$

To proceed further, we invoke a physical principle known as Torricelli's Law: For a non-viscous fluid flowing in a jet from a hole in a container, where the area of the hole is much smaller than the area of the free fluid surface, the velocity $V(t)$ of the jet at the hole is given by

$$V^2 = Kgh \tag{2}$$

where h is the height of the fluid surface above the hole, g is the acceleration due to gravity ($g =$ 981.46 cm/sec^2), and K is an "unknown" constant which applies in all such situations. Actually, K is known, and you will try to find it below. We present Torricelli's Law in the form (2) with K ambiguous, because (2) may be easily derived without experimentation, by a method called "dimensional analysis." For now, we assume (2) is true.

3) Explain why $v(t) = kV(t)$, and show that $v(t) = k\sqrt{Kgh(t))}$. Then substitute into (1) and solve for $h(t)$, to obtain

$$\sqrt{h} = -\frac{\mu t}{2} + \sqrt{h(0)} \tag{3}$$

where $\mu = k\sqrt{Kg}/A$. This equation represents our model.

III. The Payoff

1) Equation (3) predicts that $\sqrt{h}$ is a linear function of time. From your experimental data, plot v against time. Does it produce a straight line? How about for the time interval when the flow is a jet? Is this evidence that (2) is indeed valid?

2) Assuming you have a straight line in part (1), find its slope from two data points on it or use a least squares approach using all the data points. Equate this slope with the appropriate term in (3). Then using the measurements you have made, solve for K.

REFERENCES

[1] *The Journal of Irreproducible Results*. 1955. http://www.jir.com/. Accessed 15 April 2017.