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STUDENT VERSION

Space Flight For Recolonization

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STATEMENT

This is intended to be a group assignment. The primary submission is a hard copy report that should not exceed four (4) pages (not including appendices, title page, etc.). This report will include graphs reflecting your work.

The report should include any additional supporting graphs, equations, and computations in the appendices. In addition you should digitally submit any technology files.

There is also a 15 minute presentation, including time for questions, to your instructor with no more than five (5) slides (one slide per task).

The year is 2045, you are a Colonel in the Army working for U.S. Army Space and Missile Defense Command. Due to the depletion of resources on Earth and the achievement of nuclear weapons by various countries with competing interests, we have been tasked with finding a new planet to colonize in order to eventually move the United States to a new solar system, which we can fortify and defend. Before colonizing the new planet, we must obtain a better idea of the planet's orbit. In addition, we need to understand the amount of work required to travel to the new planet along with the amount of radiation exposure. We also need to develop a design for the rocket to take us to the new planet.

TASK 1: Exploring Second Order Systems and Multivariable Calculus

Let $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$ be the position vector of planet Aldrin (named after Buzz Aldrin USMA class of 1951 and former astronaut) orbiting the star named Borman (named after Frank Borman USMA class of 1950 and former astronaut). We are interested in solving Newton's Law of Gravitational

Pull, describing how any two objects exert a gravitational force of attraction on each other. The direction of the force is along the line joining the objects. The magnitude of the force is proportional to the product of the masses of the objects, and inversely proportional to the square of the distance between them known as the Inverse Square Laws (`http://theory.uwinnipeg.ca/physics/circ/node7.html`).

Therefore we can rewrite Newton's Law of Gravitational Pull as follows,

$$m\mathbf{x}'' = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}, \tag{1}$$

where,

$m =$ the mass of planet of Aldrin (6.39×10^{23} kg is the mass of Mars which we will use as an approximation to the mass of Aldrin),

$M =$ the mass of the star (4.018×10^{30} kg is the mass of Sirius which we will use as an approximation to the mass of Borman),

$G = 6.67408 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$, the Universal Graviational constant.

We can calculate,

$$\left|\frac{\mathbf{x}}{|\,\mathbf{x}\,|^3}\right| = \frac{|\,\mathbf{x}\,|}{|\,\mathbf{x}\,|^3} = \frac{1}{|\,\mathbf{x}\,|^2}, \tag{2}$$

which makes clear the name inverse square law.

We can rewrite (1) in coordinates as

$$\begin{cases} mx_1'' &= -\frac{mMG}{(x_1^2+x_2^2+x_3^2)^{3/2}}x_1, \\ mx_2'' &= -\frac{mMG}{(x_1^2+x_2^2+x_3^2)^{3/2}}x_2, \\ mx_3'' &= -\frac{mMG}{(x_1^2+x_2^2+x_3^2)^{3/2}}x_3. \end{cases} \tag{3}$$

Note that we are ignoring the gravitational pull of any other objects in the universe as well as the gravitational pull of planet Aldrin on the star Borman. We are just studying the gravitational pull of the star Borman on the planet of interest.

A solution to the system above with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}'(0) = \mathbf{v}_0$ represents the motion of a planet (star, satellite, astronaut, etc.), under the influence of gravity, that passes through the point x_0 with velocity $\mathbf{v}_0$.

i. Solve the second order system of ODEs using a numerical solver, with the initial values suggested below, to get an idea of sample solutions.

   a. $x_1(0) = 10^9, x_2(0) = 0, x_3(0) = 0, x_1'(0) = 0, x_2'(0) = .9 \times 10^5, x_3'(0) = 0$ with $0 \leq t \leq 5000$,
   b. $x_1(0) = 10^9, x_2(0) = 0, x_3(0) = 0, x_1'(0) = 0, x_2'(0) = .9 \times 10^5, x_3'(0) = 0$ with $0 \leq t \leq 500{,}000$,
   c. $x_1(0) = 10^8, x_2(0) = 0, x_3(0) = 0, x_1'(0) = 10, x_2'(0) = 10^5, x_3'(0) = 10^5$ with $0 \leq t \leq 5000$,

d. $x_1(0) = 10^8, x_2(0) = 0, x_3(0) = 0, x_1'(0) = 10^5, x_2'(0) = 0, x_3'(0) = 0$ with $0 \leq t \leq 50$,

e. $x_1(0) = 10^8, x_2(0) = 10^8, x_3(0) = 10, x_1'(0) = 10^8, x_2'(0) = 10^8, x_3'(0) = 100$ with $0 \leq t \leq 500$.

ii. Discuss the differences in solutions to initial value problems i.a., i.c., i.d., i.e., suggested above. What could these solutions represent physically?

iii. Notice the initial conditions in 1.a. gives you an ellipse, as expected, but initial conditions i.b. does not look like an ellipse. This is due to error in the numerical solver for this nonlinear, second order system. Use the $x_1(t), x_2(t)$, and $x_3(t)$ solutions from Part i.b., found using a numerical solver, to verify our hypothesis about error by computing and plotting the energy from $0 \leq t \leq 500,000$. You will see that the energy of the solution, defined as

$$E(x_1(t), x_2(t), x_3(t)) = \frac{1}{2}\left(x_1'(t)^2 + x_2'(t)^2 + x_3'(t)^2\right) - \frac{MG}{\sqrt{x_1(t)^2 + x_2(t)^2 + x_3(t)^2}},$$

is not constant for the initial conditions of Part i.b. We know by the principle of Conservation of Energy that the energy of the solution should be constant. What does this tell you about the validity of your solution using a numerical solver.

TASK 2: Finding and Parameterizing the Orbit of the Planet of Interest

The orbit of a planet must stay in a 2-dimensional plane so we may assume that our orbits are contained in the xy-plane. We also specify that the orbit should be elliptical given by the following equation in polar coordinates,

$$r(\theta) = \frac{ed}{1 + e\cos\theta}, \tag{4}$$

where e and d are parameters which represent the *eccentricity* and the *directrix* of the ellipse. Eccentricity is a parameter that determines the amount an ellipse deviates from a perfect circle. The values range from $0 \leq e \leq 1$. See Figure for different eccentricities. . Eccentricity is a parameter that determines the amount in which its orbit around another body deviates from a perfect circle[5]. The scale ranges from 0 to 1 where 1 is a perfect circle. See the image inFigure for an example of different eccentricities.[1]

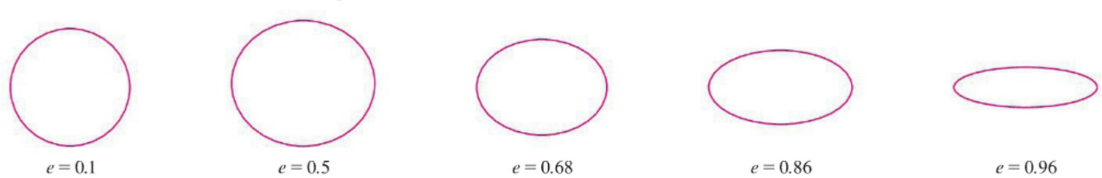

Figure 1. Eccentricities Ranging From 0 to 1 [1].

For a fixed eccentricity, the directrix controls the size of the ellipse. The value of d then should be positive and for the orbits of planets should be fairly large.

Our goal now is to find a particular curve which describes planet Aldrin by determining specific values for the parameters e and d. Our team of astronomers has collected the position of planet Aldrin's data below and our job is to find the best fit to the data given.

i. Find the best values for the parameters e and d by minimizing a SSE function. Do you need to include the z-coordinate, from Table 1, when attempting to minimize error? What is the best way for you to reconcile the data given with the model above? Can you constrain the domain over which you try to minimize SSE using physical justification relating to (3)? What is the error associated with your answer and do you think it is reasonable?

Planet Position Data	r-coordinate (meters)	θ-coordinate (radians)	z-coordinate (meters)
$\mathbf{d}_1$	2.53569×10^8	$\pi/5$	3
$\mathbf{d}_2$	3.39708×10^8	$2\pi/5$	9
$\mathbf{d}_3$	5.85633×10^8	$3\pi/5$	7
$\mathbf{d}_4$	1.41333×10^9	$4\pi/5$	-4
$\mathbf{d}_5$	3.07142×10^9	π	17
$\mathbf{d}_6$	1.41332×10^9	$6\pi/5$	-23
$\mathbf{d}_7$	5.85631×10^8	$7\pi/5$	13
$\mathbf{d}_8$	3.39718×10^8	$8\pi/5$	-3
$\mathbf{d}_9$	2.53566×10^8	$9\pi/5$	11
$\mathbf{d}_{10}$	2.31178×10^8	2π	-1

Table 1. Position of the planet Aldrin measured from the star Borman in cylindrical coordinates where $\mathbf{d}_i = (r_i, \theta_i, z_i)$.

ii. What does the gradient of the SSE represent?

iii. Kepler's Second Law says that the line joining the sun to a planet sweeps out equal areas in equal times which can be used to parameterize the motion of our planet of interest with respect to time. We can derive the following differential equation for the angle as a function of time $\theta(t)$.

$$\frac{d\theta}{dt} = \frac{h}{r^2} \tag{5}$$

where $h = \sqrt{GMed}$. Solve (5) using a numerical solver with initial condition $\theta(0) = 0$ and use your solution combined with your results Task 2 i. to find a parameterization of planet Aldrin.

TASK 3: Traveling From Our Solar System to the New World

We have been assuming that the only force we need to worry about is the force due to the star, but, now we will need to incorporate the force of the star and our Sun. Let the star be located at the origin $(0, 0, 0)$ and assume that the Sun is located at $x_0 = (10.1, 9.8, 8.7)$ light years. **(Make sure to convert all physical quantities to light years for this portion of the project.)** Then the force due to the gravitational pull of the star is given by,

$$\mathbf{F}_1 = -\frac{mM_1G}{|\mathbf{x}|^3}\mathbf{x}, \tag{6}$$

where $M_1 = 1.989 \times 10^{30}$ kg is the mass of the star Borman and $m = 2000$ kg is the mass of the spaceship. Likewise, the gravitational force of the Sun is given by,

$$\mathbf{F}_2 = -\frac{mM_2G}{|\mathbf{x}-\mathbf{x}_0|^3}(\mathbf{x}-\mathbf{x}_0), \tag{7}$$

where $M_2 = 4.018 \times 10^{30}$ kg is the mass of the Sun. So the gravitational force field of interest in order to consider the motion from Earth to planet Aldrin is given by

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2. \tag{8}$$

i. We say that a vector field $\mathbf{F}$ is conservative if there exists a function, called the potential function, such that $\nabla f = \mathbf{F}$. In the case where $\mathbf{F}$ represents a gravitational field the potential function represents the potential energy of the gravitational field. Is the vector field $\mathbf{F}$, defined above, conservative? If yes, can you find a potential function?

ii. Plot the vector field. You will have to multiply the vector field by a large scale factor in order to get a nice visual. Find the work required to travel along a straight line path under the influence of the gravitational field $\mathbf{F}$. Explore the conservative property of $\mathbf{F}$ by calculating the work along at least two different paths with the same endpoints,

$$Work = \int_\gamma \mathbf{F}\cdot\mathbf{T}ds.$$

iii. Are there any other considerations, besides gravity, which may cause you to choose a particular path?

iv. Radiation in space travel is of great concern to the health of astronauts. The closer a space shuttle approaches a star the greater the intensity of radiation. Most planets have a layer of atmosphere that provides protection from radiation but as you leave the atmosphere of a planet the protection dissipates. See Figure 2 of the Earth's exposure to radiation by the Sun.

Given the function,

$$\begin{aligned} R(x,y,z) = {} & \frac{10.4\sin^2(100x)\sin^2(100y)\sin^2(100z)}{\sqrt{x^2+y^2+z^2}} \\ & + \frac{23.5\sin^2(100(x-x_0))\sin^2(100(y-y_0))\sin^2(100(z-z_0))}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, \end{aligned}$$

which represents the radiation at a given point in space in units of Gray (Gy), calculate line integrals of radiation experienced by a rocket traveling along different paths from our solar

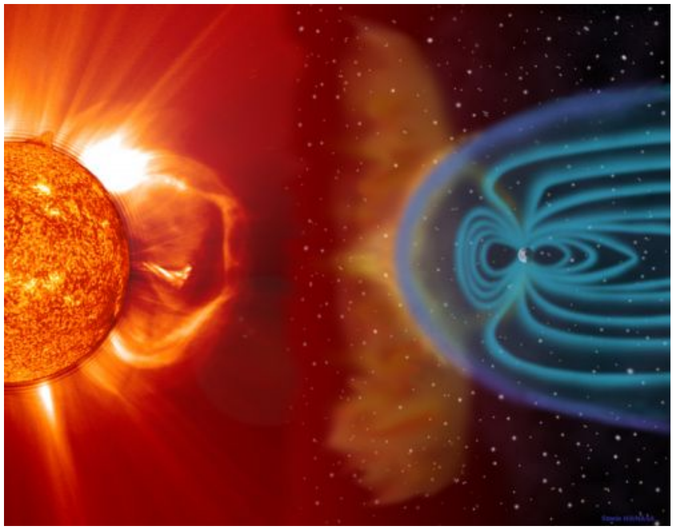

Figure 2. Earth's Exposure to Radiation [2].

Figure 3. Example of Multi-Stage Rocket [4]

system to the new solar system. Gray is a derived unit of inoizing radion does used in the International System of Units (SI). Then provide an interpretation and units analysis of these integral quantities in relation to the risk of astronauts traveling aboard the spacecraft.

We cannot graph the function $R(x, y, z)$ completely but how can we get a feel for what the function looks like visually and qualitatively?

TASK 4: Engineering the Design of the Rocket

This portion of the project is based off of a project scenario in ([1, pp. 979-980]). We significantly modified the project scenario from [1] and integrated it into the rest of the project.

The purpose of a two stage rocket is so that once the fuel is exhausted in a structure, the structure which contained the fuel is useless and only adds weight to the vehicle which slows the rocket's acceleration. By dropping the stages, which are no longer needed, the rocket lightens itself. When a stage drops off, the rocket is still traveling close to the original speed of the whole assembly.

Therefore, less total fuel is needed to reach the limits of before earth's atmosphere. In this part, we wish to minimize the total mass $M_T = M_1 + M_2$ of a two stage rocket engine, where M_1 and M_2 represent the masses of the two stages, respectively. This minimization should be carried out with the constraint that we require a final velocity, v_f, after the two stages of the rocket have been jettisoned. This will represent the typical velocity throughout our journey from earth to our new planet. If we let A represent the mass of the payload of the rocket, S the structural factor, and c the exhaust speed then we can find the following expression for the final velocity .

$$v_f = c\left[\ln\left(\frac{M_1 + M_2 + A}{SM_1 + M_2 + A}\right) + \ln\left(\frac{M_2 + A}{SM_2 + A}\right)\right].$$

The final velocity v_f is related to a concept in physics called *escape velocity* v_e, which is the velocity that an object would need to travel, away from the source of gravitation, in order to successfully escape the source of gravitation's pull on the object, without the use of thrust. The very minimum speed required, or escape velocity, can be calculated by finding the velocity such that the sum of the kinetic plus potential energy is zero ($KE + PE = 0$) [3]. The equation for this is

$$\frac{1}{2}v^2 - \frac{GM}{r} = 0.$$

Solving for escape velocity, we have,

$$v_e = \sqrt{\frac{2GM}{r}},$$

where G is the Universal Gravitational constant, $6.67408 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$, M is the mass of the planet (kg), and r is the radius of the planet (meters). The radius of the Earth, including the Earth's atmosphere, is 6.38×10^6 meters and the mass of Earth is 5.98×10^{24} kg.

i. Find the v_f in $\frac{\mathrm{m}}{\mathrm{s}}$ that is required once we pass through the atmosphere by finding the escape velocity at this point. Which of the solutions form Task 1 Part ii represents a rocket trying to escape a gravitational field?

ii. Minimize the total mass of the rocket engine, given the final velocity constraint, a structural factor, S, of 0.1, an exhaust speed of 6000 $\frac{\mathrm{mi}}{\mathrm{h}}$, and payload mass, A, of 600 pounds. Make sure your final velocity constraint is in units of $\frac{\mathrm{mi}}{\mathrm{h}}$.

iii. If we only consider the gravitational field due to earth, where we assume the earth is located at the origin for this task, we find the force field

$$\mathbf{F}_3 = -\frac{M(M_T + A)G}{|\mathbf{x}|^3}\mathbf{x},$$

which is relevant for the design of the rocket in order to escape the gravitational pull of the earth. Notice that $M_T + A$ represents the total mass of the rocket, i.e. the mass of the payload and the two stages, which is the relevant mass for the gravitational pull between the earth and the rocket, according to Newton's law, in the definition of $\mathbf{F}_3$.

Consider the path $\gamma(t) = \frac{1}{\sqrt{3}}\langle t, t, t\rangle$ on the time interval $6.371 \times 10^6 \leq t \leq 6.38 \times 10^6$ seconds which represents the rocket traveling from the earth to the radius at which we would hit the escape velocity. Remember that the particular speed of the straight line parameterization, from the surface of the earth to outside the earth's atmosphere, does not effect the calculation of work. Calculate the work due to the gravitational field $\mathbf{F}_3$ along the path $\gamma(t)$. How does this calculation of energy relate to the design of the rocket? (A discussion is fine for this last part.)

REFERENCES

[1] Stewart, James. 2015. *Calculus, 8 ed.* Stamford CT: Cengage Learning.

[2] "The Sun: Earth's Primary Energy Source The Sun and Earth's Climate Beyond Weather And The Water Cycle." Beyond Weather & The Water Cycle. `http://beyondweather.ehe.osu.edu/issue/the-sun-and-earths-climate/the-sun-earths-primary-energy-source`. Accessed 28 April 2017.

[3] "Escape Velocity" `http://hyperphysics.phy-astr.gsu.edu/hbase/vesc.html`. Accessed 28 April 2017.

[4] NASA. `https://history.nasa.gov/SP-4402/ch2.html`. Accessed 28 April 2017

[5] Orbital eccentricity. 2017. In Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Orbital_eccentricity\&oldid=766906382`. Accessed 5 May 2017.