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STUDENT VERSION
Modeling RLC Circuits

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STATEMENT

Differential equations prove exceptional at modeling electrical circuit. In fact the very simple circuit, which is fundamental to larger circuit building, and three of the most fundamental electrical object, a resistor, a capacitor, and inductor, can be modeled by just what we are doing at present, namely a constant coefficient, linear, second order differential equation. Consider the circuit in Figure 1. The EMF $E(t)$ represents an Electromotive Force generated by an excess of electrons one side of a barrier (the Switch) and a paucity of electrons on the other side of the barrier. When the switch is thrown the electrons in the excess area (say to the left of the circle marked EMF) seek to take the path of least resistance to get to the location of the paucity of electrons (to the right of the circle marked EMF). Thus we say there is a *potential* awaiting the switch to throw and that potential (just like water held high and then released to run around a descending track) causes electrons to flow clockwise through the capacitance (C), through the resistance (R), and finally through the inductance (L) until these electrons are "home" to the region of paucity of electrons.

Across each one of these device (R, L, and C) there is a change in the potential or a voltage drop and the net change in potential around the circuit clockwise starting from the left of the EMF to the right of the EMF has to equal the potential across the EMF itself. It is as if water has a drop of h feet along the route from the left of EMF, through C, through R, and then through L, back to the right side of EMF. The total drop in height over this route is the sum of the three drops over C, through R, and L. Thus h has to be equal to the sum of the drops over each of the elements, i.e., over C, over R, and over L. Call these drops h_C, h_R, and h_L, respectively, and we then note that $h = h_C + h_R + h_L$.

The electric potential is measured in a unit called a *volt* and one can think of a volt as the amount of energy required to move a unit charge to a specific spot in a static electric field. Our learning

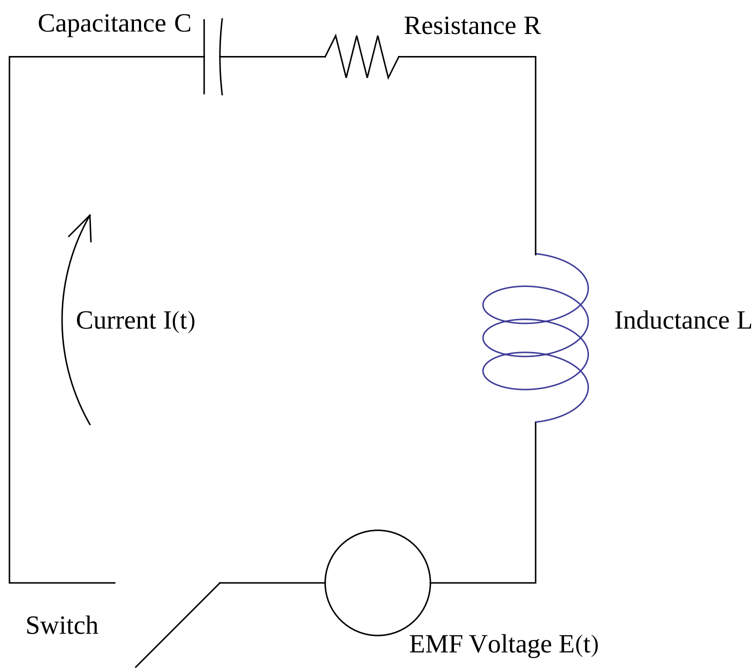

Figure 1. Diagram of an RLC circuit with imposed EMF $E(t)$ (generator or a battery) producing current, $I(t)$, in the circuit.

curve here does not include carrying all the baggage of fully developing the notions surrounding electrical circuits, even of the elementary basics of the physical devices that make up a circuit. However, we will refer to some laws, first organized by Kirchhoff and pertaining to the potential in a circuit.

Charge, $(Q(t)$, at any point in time and place is a measure of the number of electrons present and if we have 6.241×10^{18} electrons then we say we have one Coulomb of charge. If one Coulomb of charge passes by a point in one unit of time the we say we have a current. An *ampere* is a measure of the amount of electric charge (electrons) which passes by a point per unit time, or the current at a point in a circuit at time t. If some 6.241×10^{18} electrons pass a given point each second we will say there is a current flow of magnitude one ampere. Usually $I(t)$ refers to the amount of current (in amperes or amps) passing by the point or in that branch of the circuit at time t.

First we need to do some identification of the potential or voltage drop across each of the three devises we are using: capacitance (C), resistance (R), and inductance (L). Each of these devices has a unique physical construction, each is rated with different units, each deals with the electrons coming into it differently,and the effect the device has on these electrons. Each has a potential or voltage drop across it and Kirchhoff's Voltage Law says, again, that the sum of the voltages in a circuit is equal to that of the induced $\mathrm{Emf}(t)$ voltage at any time t in seconds. To get this sum we need the voltage drop across each device.

(C) The voltage (in volts) drop across a capacitor rated at C farads is

$$E_C = \frac{1}{C} \cdot Q(t)\,. \tag{1}$$

(R) The voltage (in volts) drop across a resistor rated at R ohms is

$$E_R = R \cdot \frac{dQ(t)}{dt} = R \cdot I(t)\,. \tag{2}$$

(L) The voltage (in henry) drop across an inductor rated at L henrys is

$$E_L = L \cdot \frac{dI(t)}{dt}\,. \tag{3}$$

Thus from expressions (1), (2), and (3) and the fact that these voltages have to sum up to equal the induced voltage, $\mathrm{Emf}(t)$, across the circuit, i.e.,

$$E_C + E_R + E_L = \frac{1}{C} \cdot Q(t) + R \cdot \frac{dQ(t)}{dt} + L \cdot \frac{dI(t)}{dt} = \mathrm{Emf}(t)\,. \tag{4}$$

Now watch the magic as we differentiate both sides of (4) to obtain something akin to a familiar looking (4).

$$\frac{1}{C} \cdot \frac{dQ(t)}{dt} + R \cdot \frac{dQ^2(t)}{dt^2} + L \cdot \frac{dI^2(t)}{dt^2} = \mathrm{Emf}'(t)\,. \tag{5}$$

Finally, writing the variables in terms of $I(t)$ and not $I(t)$ and $Q(t)$ we see (6)

$$\frac{1}{C} \cdot I(t) + R \cdot \frac{dI(t)}{dt} + L \cdot \frac{dI^2(t)}{dt^2} = \mathrm{Emf}'(t)\,. \tag{6}$$

Of course we will have to tell the circuit what initial current is in it, i.e, $I(0) = 0$ usually until we turn the switch on and also $I'(t) = 0$. There you have it, (6) is exactly the same as (7):

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = f(t)\,, \quad y(0) = y_0\,, \quad y'(0) = v_0\,. \tag{7}$$

If we make the identifications in Table 1 all the tools we developed for the mechanical motion situation of spring mass damper will be useful in our study of RLC circuits.

Of course, this RLC circuit equation will need a non-homogeneous term, $\mathrm{Emf}(t)$ to supply current, and as we have not studied non-homogeneous differential equations for spring mass dasher systems yet either we will need to address the solution of such equations in future activities.

Let us look at one circuit before leaving them sit for a while.

Spring Mass Dashpot	RLC Circuit
mass m	inductance L
resistance c	resistance R
spring constant k	inverse of capacitance
forcing function $f(t)$	derivative of induced voltage $\mathrm{Emf}'(t)$

Table 1. Comparison of terms between Spring Mass Dashpot and RLC Circuit differential equation.

Simple RLC Circuit Model, Solution, and Interpretation

We can, however, examine a circuit in which a current is present and does not have a driving $\mathrm{Emf}(t)$, expecting things to dampen out, in this case current to run out.

Consider an RLC circuit as depicted in Figure 1 in which we have an initial current, $I(0) = 3.2$ amps with a resistance of $R = 7$ ohms, an inductance of $L = 1$ henry, and a capacitance of $C = 0.1$ farads. Since we have some current in the circuit already $I(0) = 3.2 > 0$ at the start we shall not need an inducing $\mathrm{Emf}(t)$, so $\mathrm{Emf}(t) = 0$. Let us see what happens to the current in the circuit by solving the appropriate RLC circuit differential equation (8)

$$\frac{1}{0.1} \cdot I(t) + 7 \cdot \frac{dI(t)}{dt} + 1 \cdot \frac{dI^2(t)}{dt^2} = \mathrm{Emf}'(t) = 0\,, \quad I(0) = 3.2 \quad \text{and} \quad I'(0) = 0\,. \tag{8}$$

or as we should be accustomed to reading it

$$1 \cdot \frac{dI^2(t)}{dt^2} + 7 \cdot \frac{dI(t)}{dt} + 10 \cdot I(t) = 0\,, \quad I(0) = 37 \quad \text{and} \quad I'(0) = 0\,. \tag{9}$$

i) Solve the RLC circuit differential equation (9) for $I(t)$. Warning: *Mathematica* reserves the variable `I` as the symbol for i the imaginary square root of -1 so we will revert to using $x(t) = I(t)$ for the current in the circuit.

ii) Consider the values of R to be 0.007, 0.07, 0.7, 70, and then 700, and solve (9) in each case, keeping all other values the same, and plot the solution for the current in the circuit over the time interval $[0, 25]$ s with a vertical plot interval $[-3, 3]$. Identify each plot with its associated R value and describe what is happening to the current, $I(t)$, in each corresponding circuit over time, t. These plots are very interesting, will need some explanations, and will prove promising for future studies of electric circuits. We will stop here for now in the study of electrical circuits though.