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STUDENT VERSION
Figuring Fluid Flow

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STATEMENT

Evangelestia Torricelli (1608-1647) was an Italian physicist and mathematician who also served Galileo as a secretary. He investigated the concepts of atmospheric pressure and vacuums and he invented the barometer.

Torricelli was interested in fluids and their flow rates. He studied the following situation. Consider a cylindrical container (e.g., tin can open at top and closed at bottom) containing water. A small hole has been placed in the side of the can and the can is resting at the edge of the surface of a platform (on a table) with the hole in the can over the edge of the platform.

Now the distance from the table to the hole in the can is h_2 cm. And the water level in the can is h_1 cm above the hole in the side of the can. We can assume the diameter of the hole is small (but real) compared to heights h_1 and h_2, e.g., $h_1 = 15$ cm and $h_2 = 10$ cm, while the diameter of the hole is only 0.3 cm.

Torricelli was interested in the rate at which the water would flow out of the hole both as a function of height and time. Equivalently he was interested in the change in volume of the water in the cylinder per unit time or the change in the height, h, (for a fixed radius, r, cylinder) per unit time, i.e. he was interested in $h'(t)$ where $h(t)$ is the heighit of the ater at time t.

Certainly one could reason that with more water in the cylinder, hence more weight on the water above the hole, there would be more force pressing down, thus causing more force on the water going out. This would mean there is more water per unit time exiting the hole when h is big than when h is small.

This suggests that when h is big, $h'(t)$ ought to be big and when h is small, $h'(t)$ ought to be small. Indeed, when $h = 0$ we must have $h'(t) =$?

Could $h'(t) = -\alpha h(t)$? Could $h'(t) = -\alpha\sqrt{h(t)}$? Could $h'(t) = -\alpha h(t)^2$? Could $h'(t) = -\alpha h(t) + \beta$? Why would you not expect the latter to be the general form?

There are a number of possibilities for $h'(t)$ – the understatement of the project! What other function of h could we place on the right hand side of $h'(t) =$? What characteristics would this function have to possess? Think about the physical situation under study. Let us consider just two of these possibilities:

$$h'(t) = -\alpha h(t)\,, \tag{1}$$

and

$$h'(t) = -\alpha\sqrt{h(t)}\,, \tag{2}$$

We shall offer some data to test (1) and (2) .

We shall need a plastic jar (either translucent or transparent) of constant radius, marked with cm markings going vertically (0 being at the small exit hole in the side of the jar), a watch - accurate to the nearest second will do, some water, a collection container for when the water exits the jar, and some graph paper.

We shall fill the jar to a height of h_1 above the exit hole, uncover the hole, observe the height of the water at various intervals (try some to get a sense of how often to take data), and record that height. Now we can plot $h(t)$, the height of the water from the hole versus t on the graph paper.

If it is not possible to build an apparatus and collect data then we provide historical data in Table 1.

Activity 1

If $h'(t) = -\alpha h(t)$ (our first conjecture) then we can solve for $h(t)$ explicitly using $h(0) = h_0$ which we can set at the start of our experiment as in our data in Table **??** in the APPENDIX.

Somehow we have to find a reasonable value of α which will permit the solution of our equation, $h'(t) = -\alpha h(t)$, to fit the data. Do that now and plot the model (theoretical) equation over the data. How does it look?

Activity 2

If $h'(t) = -\alpha\sqrt{h(t)}$ (our second conjecture) then we can solve for $h(t)$ explicitly using $h(0) = h_0$ which we can set at the start of our experiment as in our data in Table 1 in the Appendix.

Again, we have to find a reasonable value of α which will permit the solution of our equation, $h'(t) = -\alpha\sqrt{h(t)}$, to fit the data. Do that now and plot the model (theoretical) equation over the data. How does it look?

Activity 3

If $h'(t) = -\alpha h(t)^2$ (our third conjecture) then we can solve for $h(t)$ explicitly using $h(0) = h_0$ which we can set at the start of our experiment as in our data in Table 1 in the Appendix.

Again, we have to find a reasonable value of α which will permit the solution of our equation, $h'(t) = -\alpha h(t)^2$, to fit the data. Do that now and plot the model (theoretical) equation over the data. How does it look?

Thus we have three conjectures, models if you will. One may appear “better” than the other. Which one appears better and why?

Write up your results so far, being sure to address the questions asked.

APPENDIX

We enter data collected in a classroom laboratory by students in 1989. The data consists of the height of water, $h(t)$, in cm in a graduated cylinder at time t in s. The data was collected by reading the times on the clock as the water passed through the given height. Notice the data is presented backwards ferom when the cylinder is empty ($t = 14$ s) until it is full ($t = 0$ s).

Height - cm	0	36.5	72.7	111.6	154.5	195.8	241.1	285.6
Time - s	14	13	12	11	10	9	8	7
Height - cm	337.4	394.4	455.3	512.9	586.1	661.3	756.8	
Time - s	6	5	4	3	2	1	0	

Table 1. Data collected on height in cm vs. time in s for water in graduated cyilinder.

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