

STUDENT VERSION

Modeling with Differential Equations

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STATEMENT

The following is material reproduced from a larger mathematics project. All internal references to course work or retained.

The world around us poses many questions that can in theory be better understood using mathematics, and the process of writing down an equation describing how a variable of interest changes according to time or space, is called mathematical modelling. Unfortunately though, we are usually not able to write down such an equation straight away. But, we can often make simplifying assumptions that allow us to write down its rate of change with time: this gives us a differential equation. Solving this differential equation then gives us the solution to the original problem. For example; if we wished to describe how the number of bacteria, y , in a particular culture grew with time, t , proceeding to immediately write down a functional form for y 's dependence upon t , $y(t)$, is no easy feat. However, if we were to make the assumption that the number of bacteria grows at a rate proportional to its current size we can write down a differential equation for y :

$$\frac{dy}{dt} = \beta y,$$

with β a constant of proportionality. We could then proceed to use integration in order to find the form for y and we're done!

Now for STEP you need to be able to formulate your own simple differential equations, to then continue to solve. You should have gained some practice in this from your A-Level modules, however a good plan to keep in mind is the following four point procedure:

1. **Identify the problems variables:** What are the dependent and independent variables we need to include?

2. **Construct the differential equation:** What is the rate of change of the dependent variable given by the problem?
3. **Solve the differential equation:** Use one of the standard methods, covered in more detail in the other articles in this module, to find the solution.
4. **Interpret the mathematical solution:** Does the solution make sense in relation to the original problem? Are there any particularly interesting elements of the solution?

One thing that the above does not mention though, that we should give greater consideration to is using initial or boundary conditions to find our exact solution of interest. Having integrated the differential equation, as usual a constant of integration will be introduced. You need to be able to interpret the original problem to write down the implied initial or boundary condition. Hopefully initial conditions will be familiar from A-Level, but boundary conditions may not. These imply a particular behaviour for a solution as the independent variable tends to some limit. For example if we integrated to find:

$$y(t) = e^{-t} + C,$$

and our original problem implies $y \rightarrow 0$ as $t \rightarrow \infty$, then necessarily we must have $C = 0$.

So the above covers the basics of formulating a differential equation, but to get good at this you need to gain much experience. Therefore, we'll now proceed to go through a few more examples before posing some problems for you to attempt!

Example 1

An object of mass $m = 5$ kg is falling towards earth, with drag proportional to its velocity with a drag coefficient of 2 kg/sec. Formulate a differential equation for the velocity v .

Solution

Here we need a little bit of knowledge from mechanics, to know that we can write down a differential equation for v using $F = m \cdot a = m \frac{dv}{dt}$. Then, all we need to do is realise the forcing pulling the object towards earth is going to be mg , and the drag slowing it down will be $2v$ using the information in the question. Putting this all together we have our differential equation:

$$m \frac{dv}{dt} = mg - 2v.$$

Example 2

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the room the object is in, and its own temperature. Write down a differential equation which would describe the temperature of the object as a function of time.

Solution

Clearly here our dependent variable will be the temperature of the object T , which depends on the independent variable t . Additionally here though we'll need a constant to describe the temperature of the room that we assume doesn't change with t . We'll call this T_r . Now we having everything we need to write down our differential equation using the information in the problem:

$$\frac{dT}{dt} = k(T - T_r).$$

Exercise 1

A drug is administered to a patient as a constant rate c . As it is administered though it is converted by the patients body to other substances at a rate proportional to its current concentration. Formulate a differential equation for how the concentration of the drug, D , changes with time, t .

Exercise 2

The growth of a particular species of algae, A , in time, t , is assumed to be jointly proportional to both \sqrt{A} and $10A$. Write down the implied differential equation for A .

Summary

OK, so that's the basics of mathematical modelling using differential equations!