SOME APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS TO REAL WORLD SYSTEM

MSc Graduate Seminar

By

Mersha Amdie Endale

May 2015

Haramaya University

SOME APPLICATION OF FIRST ORDER DIFFERENTIAL EQUATIONS TO REAL WORLD SYSTEM

A Graduate Seminar Submitted to the College of Natural and Computational Sciences, Department of Mathematics

School of Graduate Studies

HARAMAYA UNIVERSITY

In Partial Fulfillment of the Requirements for the Degree of MASTER OF SCIENCE

IN MATHEMATICS (*DIFFERENTIAL EQUATIONS*)

By

Mersha Amdie Endale

Advisor: Simegne Tafesse (PhD)

May 2015

Haramaya University

SCHOOL OF GRADUATE STUDIES

HARAMAYA UNIVERSITY

I hereby certify that I have read this Graduate Seminar entitled **“Some Applications of First Order Differential Equation to Real World System”** prepared under my direction by Mr. Mersha Amdie . I recommend that it be submitted as fulfilling the Graduate Seminar requirement.

Simegne Tafesse (PhD) ____________________ ________________

Name of Advisor Signature Date

As members of the Board of Examiners of the M. Sc. Graduate Seminar Open Defense Examination, I certify that I have read evaluated this Graduate Seminar prepared by **Mersha Amdie Endale** and examined the candidate **.** I recommended that it be accepted as fulfilling the Graduate Seminar requirement for the Degree of Master of Science in Mathematics.

________________________ ___________________ ________________

Chairperson Signature Date

________________________ ___________________ ________________

Internal Examiner Signature Date

Final approval and acceptance of the Graduate Seminar is contingent upon the submission of the final copy to the Council of Graduate Studies (CGS) through the candidate’s Department Graduate Committee (DGC).

PREFACE

The subject of differential equations is important part of mathematics for understanding the physical sciences. Most differential equations arise from problems in physics, engineering and other sciences and these equations serve as mathematical models for solving numerous problems in science and engineering.

This seminar report consists of three chapters. The first chapter deals with an introduction and preliminary concepts of the theory of differential equations that will be helpful for the main body of the seminar.

The second chapter focuses on methods of solving some differential equations particularly on solving separable , Linear first order differential equations, homogeneous equations and Bernoulli's Equation that will be used in unit three of this seminar paper for our real world system application.

Lastly, chapter three discusses on some applications of first order differential equation to real world system. Here the applications are supported by an illustrative examples.

ACKNOWLEDGMENT

My sincere appreciation extends to my advisor Dr. Simegne Tafesse, for his great help in advising, editing, providing materials as well as other required supports during the preparation of this seminar.

I would like to express my special gratitude to my friend Tekola Demisie for his encouragement and assistance during my work.

Last but not least thanks to God the Almighty for all the accomplishments.

TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION AND PRELIMINARY CONCEPTS

1.1. Introduction

Many problems in engineering and science can be formulated in terms of differential equations. The formulation of mathematical models is basically to address real-world problems which has been one of the most important aspects of applied mathematics. It is often the case that these mathematical models are formulated in terms of equations involving functions as well as their derivatives. Such equations are called differential equations. A differential equations is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. If only one independent variable is involved, the equations are called ordinary differential equations, otherwise it is called Partial differential equation.

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's laws allow to relate the position, velocity, acceleration and various forces acting on a body and state this relation as a differential equation for the unknown position of the body as a function of time.

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions as the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers.

Referring to the number of its independent variables while differential equation is divided into the types ordinary differential equations and partial differential equations, they can further described by attributes such as linearity and order. When we classify DE by linearity we have **linear** and non **linear** differential equation.

A first order differential equation is a differential equation which contains no derivatives other than the first derivative and it has an equation of the form

$$\frac{dy}{dx} = F(x, y) \text{ , where y is a function of x}$$

This seminar paper mainly focus on the application of first order differential equations to real world system which considers some linear and non linear models, such as equations with separable variables , homogeneous and Bernoulli's Equation equations with first order linear and non linear differential equations are discussed and also modeling phenomena for real world problems which are described by first-order differential equations is discussed in detail. The models includes Newton's cooling or warming law, population dynamics, Harvesting of renewable natural resources, Prey and predator and a falling object with air resistance .

1.2. Statement of the problem.

Many of the principles in science and engineering concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, such principles are often expressed in terms of differential equations. Many fundamental problems in biological, physical sciences and engineering are described by differential equations. On the other hand, physical problems have motivate the development of applied mathematics, and this is especially true for differential equations that helps to solve real world problems in the field. Thus, making a study on application of differential equation essential .

Formulating differential equation to real world problem is not easy. To formulate and use differential equation in real world system first we have to identify the real world problems that need a solution; then make some simplified assumptions and formulate a mathematical model that translate the real world problem into a set of differential equation . Next apply mathematics to get some sort of mathematical solution and then interpret the results. Thus, we are giving attentions to mathematical modeling, deriving the governing differential equations from physical principles.

In this seminar report we discussed linear and non linear first order differential equations, their solution methods and the role of these equations in modeling real-life problems. From this discussion we get some idea how differential equations are closely associated with physical applications and also how different problems in different fields of science is formulated in terms of differential equations.

Objectives.

This seminar is intended to explore the following specific objectives.

- To study some real world problems which are described by first order differential equations
- To apply Logistic growth model of population which is expressed by first order nonlinear differential equation for population growth of animals when overcrowding and competition resources are taken into consideration .
- To discuss the use of mathematical model for harvesting renewable natural resources
- To analyze and interpret some real world application problems of first order linear and non linear differential equations

Methodology

Academic activities like researches and projects works have their own methodology. Likewise this seminar work has also its own methodology which is stated below.

- The seminar is conducted by collecting some important information from different sources such as books and internet which are related to the topic.
- The facts and concepts obtained from different sources related to the topic of this seminar organized in proper manner
- Some first order differential equations of real world problems is studied.
- Some facts and concepts related to the topic of this seminar paper is discussed.

1.3. Preliminary Concepts

Definition 1.1 An equation containing the derivatives of one or more dependent variable, with respect to one or more independent variables, is said to be a differential equation (DE).

Definition 1.2 A differential equation is said to be an ordinary differential equation (ODE) if it contains only derivatives of one or more dependent variables with respect to a single independent variable. In symbols we can express an n^{th} order ordinary differential equation in one dependent variable by the general form

$$F(x, y, y', y'' \ldots y^{(n)}) = o, \tag{1.1}$$

where F denotes a mathematical expression involving $x, y, y', y'', \ldots, y^{(n-1)}, y^{(n)}$ and

where $y^{(n)} = \frac{d^n y}{dx^n}$

Definition1. 3 *A partial differential equation* is a differential equation which involves two or more variables and its partial derivative with respect to these variables.

Definition 1.4 The order of a differential equation is the order of the highest derivative in the equation.

Definition 1.5 The degree of a differential equation is the degree of the highest order derivative in the equation

Definition 1.6 First order first degree differential equation is a differential equation which contains no derivatives other than the first derivative and it has an equation of the form

$$\frac{dy}{dx} = F(x, y) \text{ , where } y \text{ is a function of } x \tag{1.2}$$

and we rewrite this equation in the form $y' = \frac{dy}{dx} = F(x, y)$

Definition 1.7 An n^{th}-order ordinary differential equation is said to be linear in y if it can be written in the form

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)} + \ldots\ldots + a_1(x)\, y' + a_o(x)y = f(x) \tag{1.3}$$

where $a_o, a_1, a_2 \ldots, a_n$ and f are functions of x on some interval, and $a_n(x) \neq 0$. The functions $a_k(x)$, k= 0, 1, 2, ... , n are called the coefficient functions.

If n=1, in equation (1.3), we get a linear first order differential equation and it can be written in the form

$a_1(x)\, y' + a_o(x)y = f(x)$, $a_1(x) \neq 0$ (1.4)

if $y' = \frac{dy}{dx}$, $\frac{a_0(x)}{a_1(x)} = p(x)$, $\frac{f(x)}{a_1(x)} = q(x)$, equation (1.4) is equivalent to

$$\frac{dy}{dx} + p(x)y = q(x)$$

Definition 1.8 A differential equation that is not linear is called non-linear.

Definition 1.9 A separable differential equation is a DE in which the dependent and independent variables can be algebraically separated on opposite sides of the equation.

Definition 1.10 A differential equation of the from $y' = f(x, y)$, $y(x_o) = y_o$ or $f(x, y, y') = 0$, $y(x_o) = y_o$ is called initial –value problem (IVP) and $y(x_o) = y_o$ is called initial condition.

A solution of the IVP (if exists) must satisfy the initial condition .

Definition 1.11: A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution**

Definition 1.12 : **A Particular Solution** is a solution obtained from a general solution by choosing particular value of the arbitrary constants.

Definition 1.13: A singular solution of a differential equation are solutions which cannot be obtained from the general solution (i.e unusual solution)

Definition 1.14 A mathematical model of a real-life situation is a set of mathematical statements which describe the situation in mathematical terms.

Picard's Theorem : Suppose that both f(x, y) and its partial derivative $\frac{\partial f}{\partial y}$ are continuous on the interior of a rectangle *R*, and that (x, y) is an interior point of *R*. Then the initial value problem

$$y' = f(x, y), \qquad y(x_o) = y_o \ \frac{\partial f}{\partial y}$$

has a unique solution $y = y(x)$ for x in some open interval contain x_o

Picard's Theorem guarantees both the existence and uniqueness of a solution

Definition 1.15 We say that the constant function $y = c$ is an equilibrium solution of the differential equation $y_o = f(t, y)$ if $f(t, c) = 0$ for all t.

Definition 1.16 An equilibrium solution is said to be **stable** if all near by solutions move towards the equilibrium solution as $t \rightarrow \infty$.

Definition 1.17 An equilibrium solution is **unstable** if solutions close to the equilibrium solution tend to get further away from that solution as $t \rightarrow \infty$.

CHAPTER TWO

SOLVING SOME FIRST ORDER DIFFERENTIAL EQUATIONS

2.1. Introduction

The solution of a differential equation may take the form of the dependent variable being expressed explicitly as a function of the independent variable y = f(x) or implicitly as in a relation of the type f(x, y) = 0. Not every differential equation has a solution. In fact, few differential equations have exat solutions. Of those that do, only a few can be solved in closed analytic form. Only a few can have a solution that can be expressed in terms of the elementary functions (i.e. the rational algebraic, trigonometric, exponential and logarithmic functions). Some others can be solved in terms of higher transcendental functions. For those which cannot be solved analytically, one can use different techniques such as power series representation and numerical approximation methods.

There is no general procedure for solving a differential equations that has a solution. Only a few simple equations can be solved by integrating directly. Most equations are solved by techniques devised for a particular type of equation. Generally, to solve a differential equation one must be able to recognize the type of the equation and use the proper method for solving it.

In this chapter, we only be concerned with first order differential equations. We consider the first order differential equations in many forms like homogeneous, linear, non linear, exact, separable variable form etc. Here we will discuss some of the types of first order differential equations that have many applications, namely separable, linear, homogeneous equations and Bernoulli's differential equations.

Before we go to the next section we state the theorem of existence and uniqueness of solutions of first order differential equation below

Theorem 2.1 (Existence and uniqueness theorem for first order DE)

Let f(x, y) be a real valued function which is continuous on the rectangle

R=$\{(x,y): |x - x_0| \leq a, |y - y_0| \leq b\}$. Assume f has a partial derivative with respect to y and that $\frac{\partial f}{\partial y}$ is also continuous on the rectangle R. There exist an interval I= $[x_{0-}h,\ x_{0+}h\,]$ (with

$h \leq a$) such that the initial value problem $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ has a unique solution y(x) defined on the interval

Next we will discuss how to solve separable variable, linear types of first order differential equations , homogeneous equations and Bernoulli's Equation . We need different approaches for solving these types of differential equations.

2.2 Separable First Order Differential Equations

Definition 1: A separable first order differential equation is a differential equation which may be put into one of the following forms:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}, \quad \frac{dy}{dx} = \frac{g(y)}{f(x)} \quad \text{or} \quad \frac{dy}{dx} = f(x)\, g(y) \qquad (2.1)$$

To solve a separable first order differential equations go through the following steps

Steps For Solving a Separable Differential Equation :

$\frac{dy}{dx}$ = F(x, y)

1. Recognize the problem as a separable differential equation .

e.g. we have $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

2. Separate the equation into the form

(terms involving y)(dy) = (terms not involving y)(dx)

e.g. rearrange $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ to g(y)dy = f(x)dx

3. Integrate both sides, adding one arbitrary constant, say C, to the x side.

e.g Evaluate $\int g(y)\, dy = \int f(x)\, dx$ + C

4. Solve for y if possible.

If we can solve for y, we have the explicit general solution, and if we cannot solve for y we have an implicit general solution.

5. If it is an initial value problem, plug in the given values and solve for a constant C.

Example 2.2.1: Solve $\frac{dy}{dx} = \frac{\cos x}{y + e^{y}}$

We see that we have a separable D.E., so we proceed as usual:

(y + e^{y}) dy = cos xdx

$\int (y + e^y)dy \;=\; \int cosx\, dx$

$\frac{y^2}{2} + \; e^y = \;$ sin x + C

Here, it would be impossible, to isolate y, so we cannot put this into y = y(x) form to get our usual explicit general form of the solution. However, the solution we have found, relating x and y, is an implicit general solution to the given D.E.

We now move on to first order linear differential equations.

2.3 First Order Linear Differential Equations.

Consider the DE $\quad \frac{dy}{dx} + p(x)y = q(x)$ (2.2)

where $p(x)$ and $q(x)$ are continuous functions of only x. This is the **standard form of first order linear** differential equation .

The approach we use for solving linear D.E.'s is very different from the one we used for separable D.E. In order to state our solution method, we need the following definition.

Definition : Given a linear D.E.

$$\frac{dy}{dx} + p(x)y = q(x)$$

we define the integrating factor μ(x) for the differential equation to be:

$$\mu(x) = e^{\int p(x)dx} \qquad (2.3)$$

Note: When we calculate $\int p(x)dx$ here, we let the integration constant C to be zero.

We multiply through the equation by the integrating factor, $\mu(x) = e^{\int p(x)dx}$ to get:

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}\, p(x)y = q(x)e^{\int p(x)dx}$$

The left hand side of this equation is the derivative of $ye^{\int p(x)dx}$.

That is,

$$\frac{d}{dx}\left[ye^{\int p(x)dx}\right] = \left(\frac{dy}{dx}\right)e^{\int p(x)dx} + \left|e^{\int p(x)dx}\left(\frac{d}{dx}\int p(x)dx\right)\right|y$$

and of course $\frac{d}{dx}\int p(x)dx \; = p(x)$

So after we multiply through the D. E. by the integrating factor, what we have is:

$$\Rightarrow \frac{d}{dx}\left[ye^{\int p(x)dx}\right] = q(x)\, e^{\int p(x)dx}$$

We now integrate both sides with respect to x on both sides of the equation.

$$\int \left[\frac{d}{dx}\left(ye^{\int p(x)dx}\right)\right]dx = \int \left(q(x)\, e^{\int p(x)dx}\right)dx$$

$$\Rightarrow ye^{\int p(x)dx} = \int q(x)e^{\int p(x)dx}dx$$

Now, we divide through by $e^{\int p(x)dx} = \mu(x)$, to isolate y:

$$y = e^{-\int p(x)dx} \int q(x)\, e^{\int p(x)dx}\, dx$$

$$\Rightarrow y = \frac{1}{\mu(x)} \int q(x)\; \mu(x)dx \qquad (2.4)$$

The equation (2.4) above is the general solution to the differential equation. We now restate the above result as a theorem with out proof.

Theorem 2.3.1. Given a linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x)$$

with integrating factor $\mu(x) = e^{\int p(x)dx}$, the general solution is given by:

$$y = \frac{1}{\mu(x)} \int q(x)\; \mu(x)dx \qquad (2.5)$$

Example 2.3.1. : Solve $\frac{dy}{dx} + \frac{2y}{x} = 5x^2$

Solution: Notice that this is a linear first order differential equation, with

$p(x) = \frac{2}{x}$ and $q(x) = 5x^2$. By the previous theorem we need only find $\mu(x)$,

and then apply the formula.

$$\int p(x)dx = \int \frac{2}{x}\, dx = 2 \int \frac{2}{x}\, dx = 2ln|x| + C$$

$$= lnx^2 + C = lnx^2$$ (since we use C = 0 here).

So the integrating factor is given by

$$\mu(x) = e^{\int p(x)dx} = e^{\ln(x^2)} = x^2$$

Substituting into (2.5) of the general solution formula we can find the general solution:

$$y = \frac{1}{\mu(x)} \int q(x)\; \mu(x)dx$$

$$y = \frac{1}{x^2} \int (5x^2)(x^2)\, dx$$

$$= \frac{1}{x^2} \int (5x^4)\, dx$$

$$= \frac{1}{x^2}\,(x^5 + C)$$

So we see the general solution is : $y = \frac{x^5}{x^2} + \frac{C}{x^2} = x^3 + \frac{C}{x^2}$

2.4 Solution by substitution

A first order differential equation can be changed into a separable DE or into a linear DE of standard form by appropriate substitution. We discuss here two classes of differential equation , one class contains homogeneous equations and the other class consists of Bernoulli's equation.

2.4.1 Homogeneous equations

A function f(x, y) of two variables is called homogeneous function of degree n if $f(tx, ty) = t^n f(x, y)$ for some real number n.

A first order linear differential equation, $M(x, y)dx + N(x, y)dy = 0$ is homogeneous if both coefficients M(x, y) and N(x, y) are homogeneous function of the same degree

Methods of solutions of homogeneous equations

Homogeneous differential equations can be solved by either substituting y= ux or x =vy, where u and v are new dependent variables. This solution will reduce the equation to separable variable first order DE

Example 2.4.1: Solve $(y^2 + yx)\,dx + x^2\,dy = 0$

Let y = ux, then dy = udx+xdu and the given equation take the form

$$(u^2x^2 + ux^2)\,dx + x^2(udx + xdu) = 0$$

$$(u^2 + 2u)dx + xdu = 0$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{du}{u(u+2)} = 0$$

Solving this separable differential equation , we get

$$\Rightarrow \quad \ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| = c$$

$$\Rightarrow \frac{x^2 u}{u+2} = c_1 \quad \text{where } c_1 = e^{2c}$$

$$\Rightarrow \quad x^2 \frac{y}{x} = c_1\left(\frac{y}{x} + 2\right)$$

$$\Rightarrow \quad x^2 \frac{y}{x} = c_1(y + 2x)$$

$$x^2 y = C_1(y + 2x)$$

2.4.2 Bernoulli's Equation

A well-known *nonlinear* equation that reduces to a linear one with an appropriate substitution is the **Bernoulli equation,** named after James Bernoulli (1654–1705).

Consider the differential equation given by

$$y' + P(x)y = Q(x)y^{n} \qquad (2.6)$$

This equation is linear if n=0 , and has separable variables if n=1,Thus, in the following development, assume that n $\neq 0$ and n $\neq 1$. Begin by multiplying by y^{-n} and (1– n) to obtain

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

$$\frac{d}{dx}[y^{1-n}] + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

which is a linear equation in the variable y^{1-n} Letting z =y^{1-n} produces the linear equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

Finally, the general solution of the Bernoulli equation is

$$y^{1-n}e^{\int(1-n)p(x)dx} = \int(1-n)Q(x)\,e^{\int(1-n)p(x)dx}\,dx + C$$

CHAPTER THREE

SOME APPLICATION OF FIRST ORDER DIFFERENTIAL EQUATION

3.1. Introduction:

There are many applications of first order differential equations to real world problems. In this chapter we will discuss the following linear and non linear models as an application:

- Cooling/ and warming law
- Population growth and decay
- Harvesting renewable natural resources
- Prey and predator
- A Falling object with air resistance

3.2 Newton's law of Cooling/Warming

Temperature difference in any situation results from heat energy flow into a system or heat energy flow from a system to surroundings. The former leads to heating whereas latter leads to cooling of an object.

Imagine that you are really hungry and in one minute the cake that you are cooking in the oven will be finished and ready to eat. But it is going to be very hot coming out of the oven. How long will it take for the cake, which is in an oven heated to 450 degrees Fahrenheit, to cool down to a temperature comfortable enough to eat and enjoy without burning your mouth? Have you ever wondered how forensic examiners can provide detectives with a time of death (or at least an approximation of the time of death) based on the temperature of the body when it was first discovered? All of these situations have answers because of Newton's Law of Cooling or warming. The general idea is that over time an object will cool down or heat up to the temperature of its surroundings.

Newton's empirical law of cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and that of the temperature of the surrounding medium, the so called ambient temperature. Let $T(t)$ be the temperature of a body and let T_A denote the constant temperature of the surrounding medium. Then the rate at which the body cools denoted by $\frac{dT\ (t)}{dt}$ is proportional to T(t) - T_A .

This means that

$$\frac{dT}{dt} = k(\mathrm{T(t)} - \mathrm{T_A}) \tag{3.1}$$

where, k is a constant of proportionality . The value of the constant k is determined by the physical characteristics of the object. If the object is large and well-insulated then it loses or gains heat slowly and the constant **k** is small. If the object is small and poorly-insulated then it loses or gains heat more quickly and the constant **k** is large.

We assume the body is cooling, then the temperature of the body is decreasing and losing heat energy to the surrounding. Then we have T >T_A .Thus $\frac{dT}{dt}$ < 0 Hence the constant k must be negative.

If the body is heating , then the temperature of the body is increasing and gain heat energy from the surrounding and T < T_A . Thus $\frac{dT}{dt}$ > 0 and the constant k must be negative.

$\frac{dT}{dt}$ is the product of two negatives and it is positive

Under Newton's law of cooling we can Predict how long it takes for a hot object to cool down at a certain temperature. Moreover, we can tell us how fast the hot water in pipes cools off and it tells us how fast a water heater cools down if you turn off the breaker and also it helps to indicate the time of death given the probable body temperature at the time of death and current body temperature.

The Newton's Law of Cooling leads to the classic equation of exponential decay over time which can be applied to many phenomena in science and engineering including the decay in radioactivity.

The mathematical formulation of Newton's empirical law of cooling of an object described by $\frac{dT}{dt} = k(T - T_A)$ is a linear first-order differential equation. Since it is a simple separable differential equation, its solution is obtained by the method of separation of variables as follows

$$\frac{dT}{dt} = k(T - T_A)$$

$$\frac{dT}{(T - T_A)} = kdt$$

$\Rightarrow$ $\ln|T - T_A| = kt + c_1$, where c_1 is a constant

$\Rightarrow$ $|T - T_A| = e^{kt+c_1}$

Hence $T(t) = T_A + c_2e^{kt}$, where c_2 is a constant (3.2)

When the ambient temperature T_A is constant the solution of this differential equation is

$$T(t) = T_A + c_2e^{kt} ,$$

(This equation represents Newton's law of cooling) .

If $k < 0$, $\lim t \dashrightarrow \infty$, $e^{kt} = 0$,

This shows that the temperature of the body approaches that of its surroundings as time goes.

For the initial value problem $T(t) = T_A + c_2e^{kt}$, $T(0) = T_o$

$$T(0) = T_o = T_A + c_2e^{k0} \text{ and } c_2 = T_o - T_A$$

$$T(t) = T_A + (T_o - T_o)\, e^{kt}$$

The graph drawn between the temperature of the body and time is known as cooling curve (as shown in figure 3.1 below). The slope of the tangent to the curve at any point gives the rate of fall of temperature.

Suppose that a cup of tea starts out at time **t = 0** at a temperature of 120 degrees Celsius in a room whose temperature is 60 degrees Celsius and **k = J 0.05**. This situation can be described by the initial value problem.

$$\frac{dT}{dt} = -0.5\,(T - 60) ,\ T(0) = 80$$

The function described by this initial value problem is shown in the graph below.

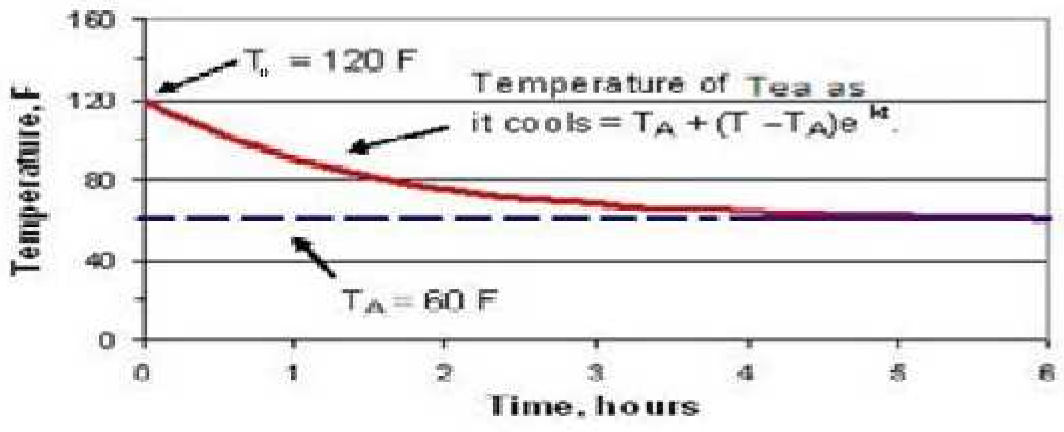


Figure 3.1

Where,

$$T(t) = T_A + (T_o - T_A)\, e^{kt} = 60 + 60\, e^{-0.05t}$$

$T(t)$ = Temperature at time t and

T_A = Ambient temperature (temperature of surroundings),

T_o = Temperature of tea at time 0,

k = negative constant and

t = time

Example 3.2.1 : When a chicken is removed from an oven, its temperature is measured at 300^0F. Three minutes later its temperature is 200^o F. How long will it take for the chicken to cool off to a room temperature of 70^oF.

Solution: In (3.2) we put T_A = 70 and T=300 at for t=0.

$T(0)=300=70+c_2e^{k.0}$

This gives $c_2=230$

For t=3, T(3)=200

Now we put t=3, T(3)=200 and $c_2=230$ in (3.2) then

$200=70 + 230\, e^{k.3}$

or $e^{3k} = \frac{130}{230}$

or $3k = \ln \frac{13}{23}$

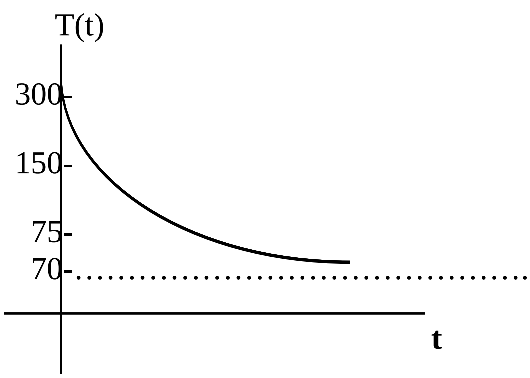
or $k = \frac{1}{3}\ln \frac{13}{23} = -0.19018$

Thus $T(t)=70+230\, e^{-0.19018t}$ (3.3)

We observe that (3.3) furnishes no finite solution to T(t)=70 since

$$\lim_{t\to \infty} T(t) = 70.$$

The temperature variation is shown graphically in the figure below. We observe that the limiting temperature is 70^0F.



Example 3.2.2: The big pot of soup as part of Jim's summer job at a restaurant, Jim learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. He also found out that , while refrigerate ration was essential to preserve the soup over night, the soup is to hot to be put directly into the fridge when it was ready (The soup had just boiled at 100 degree centigrade , and the fridge was not powerful enough to accommodate a big pot to soup if it was any warmer than 20 ^{0}C) Jim discovered that by cooling the pot in a sink full of cold water , (kept running , so that its temperature was roughly constant at 5 ^{0}C) and stirring occasionally, he could bring the temperature of the soup to 60 ^{0}C in 10 minutes. How long before closing time should the soup be ready so that Jim could put it in the fridge and leave on time?

Solution: Let us summarize the information briefly and define notation for the problem. Let

T(t) = temperature of the soup at time t (in minute)

$T(0) = T_o$ = Initial temperature of the soup=100 ^{o}C

T_A = ambient temperature (temperature of water in sink) = 5^{o}C

By Newtons law of cooling $\frac{dT}{dt}$ proportional to the difference between the temperature of the soup T(t) and ambient temperatures this means that $\frac{dT}{dt}$ is proportional to ($T(t) - T_A$). Clearly if the soup is hotter than the water in the sink $T(t) - T_A > 0$, then the soup is cool down which means that the derivative $\frac{dT}{dt}$ should be negative .This means that the equation we need has to have the following sign pattern

$\frac{dT}{dt} = -K(T(t) - T_A)$ where K is positive constant

This equation is first order differential equation. The independent variable is time t, the function we want to find is T(t) and the quantity T_A, K are constants. In fact, from Jims measurements we know that T_A =5, but we still do not know what value to put in for the constant K. Lets define the following new variable

y(t) = T(t) – T_A= Temperature difference between the soup and water in the sink at time t .

$y_0 = T(0) - T_A =$ Initial temperature difference at time t=0

Note that it if we take the derivatives of y(t), and use the Newton's law of cooling, we arrived at

$$\frac{dy}{dt} = \frac{d}{dt}(T(t) - T_A) = \frac{dT(t)}{dt} - \frac{dT_A}{dt} = \frac{dT}{dt} = -k(y) \qquad (3.4)$$

We have used the fact that T_A is a constant to eliminate its derivative, and we plugged in y for $(T(t) - T_A)$ in (3.4) . By defining the new variable, we have arrived once more at the following equation

$$\frac{dy}{dt} = -ky \quad \text{whose solution is} \quad y(t) = y_{oe^{-kt}}$$

We can use this result to conclude (by plugging in y =(T- T_A) and (T_0 - T_A) that

$$T(t) - T_A = (T_0 - T_A)e^{-kt}$$

$$T(t) = T_A + (T_0 - T_A)e^{-kt}$$

How the soup will cool ?

From the information in the problem, we know that

T(0) = $T_0 = 100$, T_A= 5, $T_0 - T_A = 95$

$T(t) = 5 + 95e^{-kt}$

The following figure shows how the soup cools with time t

$T(t) = 5 - 95e^{-0.054t}$

We also know that after 10 minutes, the soup cools to 60 ºC, so that t=10, T(10) = 60. Plugging into the last equation, we find that

60= T(10) =5+ 95e^{-10k}

Rearranging,

55= 95e^{-10k} , $\frac{55}{95} = e^{-10k}$ take the reciprocal of both sides of the equation

$$e^{10k} = 1.73$$

Taking the natural logarithm of both sides, and solving for k , we find that

$$\ln e^{10k} = \ln 1.73$$

$$k = \frac{\ln(1.73)}{10} = \frac{0.54}{10} = 0.054 \; per\ min$$

So we see that the constant which governs the rate of cooling is k = 0.054 per minute

K =0.054⇒$T(t)$ = 5 + 95$e^{-0.054t}$

The temperature of the pot of soup at time t will be $T(t)$ = 5 + 95$e^{-0.054t}$

To finish our work , let us determine how long it takes for the soup to be cool enough to put into the refrigerator. We need to wait until T(t) = 20, so at that time:

$$20 = 5 + 95e^{-0.054t}$$

$$\frac{15}{95} = e^{-0.054t}$$

$\frac{95}{16} = 0.6333 = e^{0.054t}$ take the reciprocal of both sides of the equation

Taking the logarithms of both sides , we find that

$$\ln (6.33) = \ln(e^{0.054t}) = 0.054t$$

Thus using the fact that ln(6.33) = 1.84

$$t = \frac{1.84}{0.054} = 34.18$$

Thus, it will take a little over half an hour for Jim's soup to cool off enough to be put into the refrigerator.

3.3 Population Growth

In order to illustrate the use of differential equations with regard to population problems we consider the easiest mathematical model offered to govern the population dynamics of a certain species. One of the earliest attempts to model human population growth by means of mathematics was by the English economist Thomas Malthus in 1798. Essentially, the idea of the Malthusian model is the assumption that the rate at which a population of a country grows at a certain time is proportional to the total population of the country at that time . In mathematical terms, if P(t) denotes the total population at time t, then this assumption can be expressed as

$$\frac{dP}{dt} = kP(t) \tag{3.1}$$

where k is called the growth constant or the decay constant, as appropriate.

Solution of equation (4.1) will provide population at any future time t. This simple model which does not take many factors into account (immigration and emigration, for example) that can influence human populations to either grow or decline, nevertheless turned out to be fairly accurate in predicting the population.

The differential equation $\frac{dP(t)}{dt} = kP(t)$, where P(t) denotes population at time t and k is a constant of proportionality, serves as a model for population growth and decay of insects, animals and human population at certain places and duration.

It is fairly easy to see that if $k > 0$, we have growth, and if $k < 0$, we have decay.

Equation (3.1) is a linear differential equation which solve into

$$P(t) = P_0 e^{kt} \tag{3.2}$$

where p_0 is the initial population, i.e. p(0) = p_0, and k is called the growth or the decay constant. Therefore, we conclude the following:

- if k>0, then the population grows and continues to expand to infinity, that is, $\lim_{t\to\infty} P(t) = \infty$
- if k<0, then the population will shrink and tend to 0. In other words we are facing extinction.

Clearly, the first case, k>0, is not adequate and the model can be dropped. The main argument is that has to do with environmental limitations. The complication is that population growth is eventually limited by some factor, usually one from among many essential resources. When a population is far from its limits of growth it can grow exponentially. However, when nearing its limits the population size can fluctuate, even chaotically. Another model was proposed to remedy this weakness in the exponential model. It is called the **logistic model** (also called **Verhulst-Pearl model**). The differential equation for this model is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \tag{3.3}$$

- where M is a limiting size for the population (also called the **carrying capacity).** It is the magnitude of a population an environment can support.

Clearly, when P is small compared to M, the equation reduces to the exponential one. In order to solve this equation we recognize a nonlinear equation which is separable. The constant solutions are P=0 and P=M. The non-constant solutions may obtained by separating the variables

$$\frac{dp}{P\left(1-\frac{P}{M}\right)} = kdt \ ,$$

and integration

$$\int \frac{dp}{P\left(1-\frac{P}{M}\right)} = \int kdt$$

The partial fraction techniques gives

$$\int \frac{dp}{P\left(1-\frac{P}{M}\right)} = \int \left(\frac{1}{P} + \frac{1/M}{1-P/M}\right) dP \ ,$$

which gives

$$\ln|P| - \ln\left|1 - \frac{P}{M}\right| = kt + c$$

$$\Rightarrow \qquad \frac{P}{1-P/M} = Ce^{kt} ,$$

where *C* is a constant. Solving for *P*, we get

$$P = \frac{MCe^{kt}}{M+Ce^{kt}}$$

If we consider the initial condition $P(0) = p_o$ (assuming that p_o is not equal to both 0 or *M*), we get

$$C = \frac{P_0 M}{M-P_0}$$

which, once substituted into the expression for *P*(*t*) and simplified, we find

$$P(t) = \frac{MP_0}{P_0+(M-P_0)e^{-kt}} \qquad (3.4)$$

Lets examine this solution to find out what happens to the population as $t \to \infty$: does it die out? Does it persist? Does it grow forever?

- If $P_o = 0$, then $\lim_{t\to\infty} P(t) = 0$. This implies there no population to start with.
- If $P_o = M$, then $\lim_{t\to\infty} P(t) = M$. Apparently M is that population level that is in perfect balance with its surroundings .There is no growth or decline of the population.
- If $0 < P_o < M$, $\lim_{t\to\infty} P(t) = M$. This implies that if we start with a small population (i.e., less than M), the population grows towards the balance population P=M.
- If $P_o > M$, then, $\lim_{t\to\infty} P(t) = M$. If we start with a population that is too large to be sustained by the available resources, the population decreases towards the balance population P=M .

Figure3.3.1 shows the Logistic curve .The lines P=M/2 and P=M divide the first quadrant of the *tP*-plane into horizontal bands . We know how the solution curves rise and fall, and how they bend as time passes. The equilibrium lines P=0 and P=M are both population curves. Population curves crossing the line P=M/2 have an inflection point there, giving them a **sigmoid** shape . Figure 3.3.1 displays typical population curves.

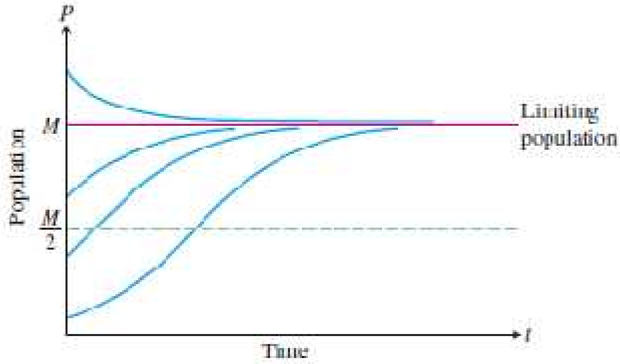


Figure 3.3.1 the Logistic solution curve

However, this is still not satisfactory because this model does not tell us when a population is facing extinction since it never implies that. Even starting with a small population it will always tend to the carrying capacity *M*.

Example 3.3.1: Let *P*(*t*) be the population of a certain animal species. Assume that *P*(*t*) satisfies the logistic growth equation

$$\frac{dP}{dt} = 0.2\, P(t)\left(1 - \frac{P(t)}{200}\right),\ P(0) = 150$$

solve this logistic growth equation

Solution:

The equation is non linear first order DE because of the presence of P^2 and separable equation.
Clearly, we have

M = the carrying capacity = $\lim_{t\to\infty} P(t) = 200$,

Let us solve by technique of solving separable equations.
First, we look for the constant solutions (equilibrium points or critical points). We have

$0.2\, P(t)\left(1 - \frac{P(t)}{200}\right),\ P(0) = 150$ if and only if P= 0 or P=200

Then, the non-constant solutions can be generated by separating the variables and the integration

$$\frac{dP}{P\left(1 - \frac{P}{200}\right)} = 0.2dt$$

$$\int \frac{dP}{P\left(1-\frac{P}{200}\right)} = \int 0.2\, dt$$

Next, the left hand-side can be handled by using the technique of integration of rational functions. We get

$$\int \frac{dP}{P\left(1-\frac{P}{200}\right)} = \int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP\ ,$$

Which gives

$$\int \frac{dP}{P\left(1-\frac{P}{200}\right)} = \ ln\left|\frac{P}{200-P}\right|$$

Hence, we have $ln\left|\frac{P}{200-P}\right| = 0.2t + C$

$$\left|\frac{P}{200-P}\right| = e^{0.2t+C} = \ Ce^{0.2\,t}$$

Where $C = \ e^{C}$

$$\frac{P}{P-200} = Ce^{0.2\,t}$$

$$P = 200\,Ce^{0.2\,t} - \text{P}\,Ce^{0.2\,t}$$

$$P + \text{P}\,Ce^{0.2\,t} = 200\,Ce^{0.2\,t}$$

$$P(1 + \ Ce^{0.2\,t}) = 200\,Ce^{0.2\,t}$$

$$P = \frac{200Ce^{0.2\,t}}{1+\ Ce^{0.2\,t}} \qquad (3.5)$$

Using $P_o = P(0) = 150$ and (3.5)

we get C = 3

when we substitute 3 for C in equation (3.5) we get

$$P(t) = \frac{600e^{0.2\,t}}{1+\ 3e^{0.2\,t}} \quad \text{or} \quad P(t) = \frac{e^{0.2\,t}}{\frac{1}{600}+\ \frac{1}{200}\,e^{0.2\,t}}$$

Therefore all solutions are

$$\begin{cases} P = 0, \; P = 200 \\ P(t) = \frac{600e^{0.2t}}{1+3e^{0.2t}} \end{cases}$$

Example 3.3.2:

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is determined that the rate at which the virus spreads is proportional not only to the number P(t) of students infected but also to the number of students not infected. Determine the number of infected students after 6 days given that the number of infected students after 4 days is 50.

Solution.

We first must find a formula for infected students P(t) which is the solution to the IVP

$$\frac{dP}{dt} = r(1000 - P)P, \qquad P(0) = 1$$

This equation can be rewritten in the form

$$\frac{dP}{dt} = 1000\,r\left(1 - \frac{P}{1000}\right)P$$

From equation (4.4) above $P(t) = \frac{MP(0)}{P(0)+(M-p(0))e^{-rt}}$

When we substitute for $P(0) = 1$ and M =1000 , we get

$$P(t) = \frac{1000}{1+999e^{-1000rt}}$$

But P(4) = 50 so that

$$50 = \frac{1000}{1+999e^{-4000r}}$$

Solving this equation for r we find r ≈ 0.0009906: Thus,

$$P(t) = \frac{1000}{1+999e^{-0.9906t}}$$

Finally, $P(6) = \frac{1000}{1+999e^{-0.9906(6)}}$ ≈ 276 students

3.4 Harvesting of renewable natural resources

Renewable resources are natural resources that can reproduce, grow, and die and nonrenewable resources are resources in which a fixed stock is depleted over time. There are many renewable natural resources that humans desire to use. Examples are fishes in rivers and sea , trees from our forests and ground water fed by rainfall. It is desirable that a policy be developed that will allow a maximal harvest of a renewable natural resource and yet not deplete that resource below a sustainable level. We introduce a mathematical model providing some insights into the management of renewable resources. The simple mathematical model develop here provides some insights into the planning process.

If P(t) represent the population at time t (measure in years) and $\frac{dP}{dt}$ is rate at which a population grows at a certain time, then the logistic equation is

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}\ P\right) = F(P) \qquad (4.1\ )$$

Where r > 0 is the growth rate and K is the environmental carrying capacity (also known as the saturated level or the limiting population since the population approaches this value over time). The value of this constant are determined experimentally.

rate at which a population of a country grows at a certain time

Without human intervention(without harvesting renewable natural resources) , the population would behave logistically depending on equation (4.1)

In this model we will further assume that human will be harvesting from animal population. The effects of harvesting a renewable natural resource such as fish can be modeled using a modification of the logistic equation (4.1) .

$$\frac{dP}{dt} = Pr\left(1 - \frac{P}{K}\right) - H(P) \qquad (4.2)$$

The first term on the right-hand side of the DE in (4.2) is a model of the population growth with a growth rate $r > 0$ and a carrying capacity (maximum sustainable population) of $K > 0$. The second term $H(P)$ is the *harvesting* term. Here we consider two different forms of $H(P)$ that leads as to constant harvesting and proportional harvesting models.

i**) Constant harvest model**

If harvesting occurs at a constant rate of h , $H(P) = h$, *equation* (5.2) becomes

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}\ P\right) - h\ = F(P) - h = G(P) \qquad (4.3)$$

The DE in (5.3) is called the **constant-harvest model**

Note that the function $G(P) = \left(\frac{-r}{K}\right) P^2 + rP - h$ is a quadratic polynomial in P whose graph is concave down . In the normal situation where the harvest rate is not too high (that is, h < max F(P) = $F(\frac{1}{2}K) = \frac{1}{4}rK$), the function G has two real Zeros on the interval $[0, K]$ .The values of the two zeros P_1 and P_2, as shown in the figure 1,are found from the quadratic formula

$$P_{1,2} = \frac{K \pm \sqrt{K^2 - \frac{4Kh}{r}}}{2},$$

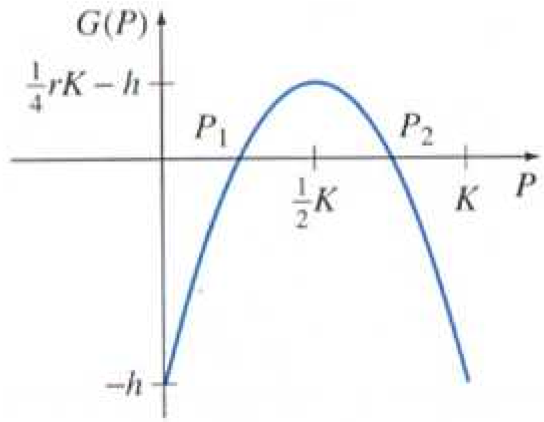


Figure (3.4.1) The logistic growth curve

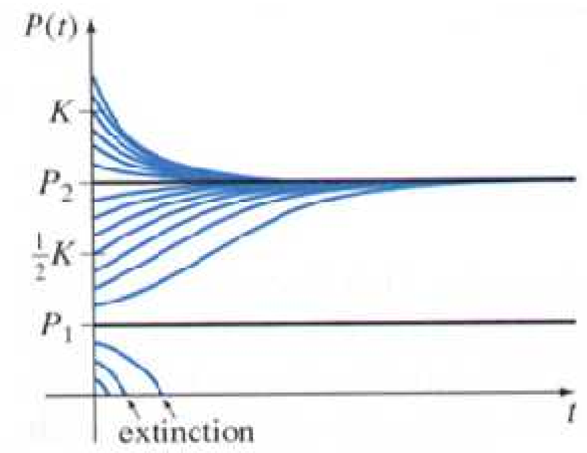


Figure (3.4.2) The logistic solution curve

We observe that that P(t) = P_1 and P(t) = P_2 are constant solutions , called equilibrium solutions of (3.4.3). Now from figure (3.4.1) we see that the derivative $\frac{dP}{dt}$ is positive on the interval $P_1 < P < P_2$, and so P will increase on the interval , while $\frac{dP}{dt}$ is negative or all other values of P . In addition by differentiating (5.3) with respect to time again , we get

$$\frac{d^2P}{dt^2} = r\left(1 - \frac{2}{K}P\right)\frac{dP}{dt} = r\left(1 - \frac{2}{K}P\right)G(P),$$

Which implies that the graph of P(t) is concave downward for P < P_1 and for $\frac{1}{2}K < P < P_2$ and concave upward otherwise. Note that if the initial population P_o less than P_1, the population P(t) decreases to zero (extinction) in finite time, otherwise the population P(t) approaches P_2, a value less than K(the limiting population without harvesting). Mathematician refer to the number P_2 as asymptotically stable equilibrium or an attractor, since other solutions that start close to P_2 approach the line P=P_2 ; the number P_1 is called unstable equilibrium or repeller. Thus we can conclude that the harvest cannot be too large without depleting the resource.

We now ask the next obvious question : How large can the harvest be and still allow a sustainable(that is, long term) harvest ? We observe earlier that there are two real solutions to G(P) =0 if h < $\frac{1}{4}$ rK. On the other hand, if h > $\frac{1}{4}$ rK, or $\frac{1}{4}$ rK-h < 0, it can be seen from figure (5.1), that $\frac{dP}{dt}$ = G(P) < 0 and so P(t) will decrease to zero. Finally, for h = $\frac{1}{4}$ rK the equation G(P)=0 has single root $P_1 = \frac{1}{2}K$ .This value of P is also a constant solution of the differential equation . The value h = $\frac{1}{4}$ rK is called the maximum sustainable yield (MSY). It is allows for a constant population of $P_1 = \frac{1}{2}K$ and a constant harvest equal to the MSY. In other words the

MSY is equal to the population added annually due to reproduction minus death.

The value of r and K may be known only to with in an accuracy of 10%. The value h = $\frac{1}{4}$ rK calculated for the MSY might, in fact, be too large for the given population , resulting in a decline to extinction.

The preceding mathematical analysis reflects the view point of most biologist who consider a resource to be overexploited when the population size has been reduced to a level below the population level at MSY

So far, we have deduced much qualitative information about solutions of (4.2) without actually solving differential equation. A solution of (4.3) subject to initial condition P(0) = P_0 can be found by separation of variable. Since P_1 and P_2 are zeros of G(P) , P-P_1 and P - P_2 must be factors of G(P). Writing $\frac{dP}{dt} = -\frac{r}{K}\,(P-P_1)(P-P_2)$ as $\frac{dP}{(P-P_2)(P-P_1)} = -\frac{r}{K}\,dt$,

Using partial fraction, and then integrating yields $\frac{1}{P_2-P_1}$ ln $\left|\frac{P-P_2}{P-P_1}\right| = -\frac{r}{K}\,t + c$ (4.4)

By applying P(0) = P_0 to (5.4) and solving for P(t), we arrive at

$$P(t) = \frac{P_2(P_0-P_1)-P_1(P_0-P_2)e^{-\alpha t}}{P_0-P_1-(P_0-P_2)e^{-\alpha t}}\ , \qquad (4.5)$$

Where $\alpha = \frac{r(P_2-P_1)}{K} = r\sqrt{1-\frac{4h}{Kr}}$. It should be clear from the explicit solution (4.5) that P(t) approaches P_2 as time t increases.

ii) Proportional harvesting model

In our next model , we assume that the harvest is proportional to the size of the population . The constant-harvest model (4.3) corresponds to a constant harvest, regardless of the time required. In the next model we assume that harvesting is proportional to the population present. In other words, instead of harvesting the same number of population each year, only a fraction of the present population to be harvested. In this scenario we write $H(P) = EP, 0 \le E < 1$,

where E > 0 is a constant referred to as the effort , and the modification of equation (4.1) becomes

$$\frac{dP}{dt} = F(P) - EP = P\left(r - \frac{r}{k}\ P\right) - EP = G(P) \qquad (4.6)$$

The differential equation in (5.6) is called the **proportional harvesting model.**

Since E is the measure of the effort that goes into harvesting resource. As before, we consider the equilibrium solutions obtained from the equation G(P) = P(r - E- $\frac{rP}{K}$) = 0. We find one positive solution $P_1 = K\left(1 - \frac{E}{r}\right)$ as long as the effort E does not exceed the growth rate r. Since (5.6) can be written $\frac{dP}{dt} = -\frac{r}{K}\,(P - P_1)$, we see that if $0 < P < P_1$ then $\frac{dP}{dt} > 0$, while if $P > P_1$ then $\frac{dP}{dt} < 0$. This indicates that a solution will always approach the equilibrium solution P(t) = P_1, making P_1 an asymptotically stable equilibrium, or an attractor. The equilibrium harvest , or **sustainable yield**, in this case

EP_1= KE$\left(1 - \frac{E}{r}\right)$. Note that the left-hand side of this last expression is quadratically dependent upon E(the right hand side) and has a maximum when E = $\frac{1}{2}r$ and $P_1 = \frac{1}{2}K$. For these latter values the number EP_1 is the maximum sustainable yield, or MSY.

To conclude this model, we note that we can solve (5.6) analytically. In fact, equation (4.6) in the form

$$\frac{dP}{dt} = P\left(r - E - \frac{r}{k}\;P\right) \tag{4.7}$$

Solving equation (4.7) gives

$$P(t) = \frac{(r-E)P_0}{\frac{rP_0}{K} + \left(r - E - \frac{rP_0}{K}\right)e^{-(r-E)t}} \tag{4.8}$$

It is seen from (5.8) that the limiting population as $t \to \infty$ is $K\left(1 - \frac{E}{r}\right)$

Example 3.4.1:

As an example, Let as estimates the value r = 0.08 and K=400000 for the certain fin whale, with P_o =P(0) =70,000

(a) In the constant harvesting model, we know that the MSY is given by h = $\frac{1}{4}$ rK=8000, with a fixed population (the population that a MSY allowed) of $P_1 = \frac{1}{2}$ K = 200,000. However, since the initial population $P_o = 70000 < P_1$, the population will decline to zero. Since the reproduction of the first year (population times growth rate =70,000 x0.08 =5600) is less than the harvest h=8,000 the population still decline to zero.

(b) For the constant effort model, let us assume that the effort is one-half the growth rate in part (a) that is, E = $\frac{1}{2}r$ = 0.04. Then $P_1 = \frac{1}{2}\;K = 200{,}000$, and the first year harvest is EP(0) =0.04 x 70,000 =2800. The MSY is EP_1=0.04 x 200,000 =8000.

Example 3.4.2

The data for this example is has been obtained from the Department of Fisheries of Malaysia and from the fish owner of selected ponds suggested by the Department of Fisheries of Malaysia situated at Gombak, Selangor, Malaysia. The Department of Fisheries Malaysia (2008) claimed that a fish pond can sustain 5 tilapia fish for every 1m^2 surface area. The selected pond has an area of 156100 m^2, the sustainable or carrying capacity, *K* of the pond is 780500 fish. The period of maturity for the tilapia fish is 6 months and estimates that 80% will survive to maturity (Thomas & Michael 1999). The Logistic Growth model can be written as:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right)$$

Here the variable *P* can be interpreted as the size of the population. *P*(*t*) depends on its initial value *P*(0) and on the two parameters *r* and *K*, where *r* is called the rate of fish survive at maturity stage and *K* is referred to as the carrying capacity of the population. When Parameter *H* was introduced as harvesting function, the logistic growth model with constant harvesting is as follows;

$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) - H(t)$, where the value of *H* is constant

The value of of the parameters are r = 0.8, the estimation of fish that will survive at maturity stage and the value of carrying capacity, K = 780500. The equilibrium point is also called critical point or stationary point. At this critical point the fish population remains unchanged.

To determine the equilibrium points for *H* constant:

$$0.8P\left(1 - \frac{P}{780500}\right) - H = 0$$

$$0.8P - \frac{0.8P}{780500} - H = 0$$

$0.8P - 0.00000102498P^2 - H = 0.$

By using square quadratic formula:

$a = 0.00000102498$, $b = -0.8$, $c = H$

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$P = \frac{-0.8 \pm \sqrt{(0.8)^2 - 4(0.00000102498)H}}{2(\,0.00000102498)}$$

Letting this expression equals to 0, we have:

$(0.8)^2 - 4(0.00000102498)H = 0$

$0.64 - 0.00000409992H = 0$

$H = 156100.6068 \cong 156100$

When the value of H = 156100 , then we consider 3 values of harvesting:

1. H = 156100

2. H > 156100

3. H < 156100

For H = 156100

From Figure 3.4.3, the value of harvesting, we have one equilibrium point. For P_o larger than 389482; the population will decrease and approach to 389482. For P_o less than 389482; the population will lead to extinction.

Table 2 shows the interval of equilibrium point that shows whether the equilibrium is stable or unstable point.

For H > 156100

From Figure 3.4.4, the value of harvesting is H = 160000 and this figure shows the decreasing trends of tilapia fish population. This implies the fish population will go to extinction regardless of the initial population size.

For H < 156100

From Figure 3.4.5 , the value of harvesting, H = 140000. There are two equilibrium points exist when the value of harvesting is less than 156100. The upper equilibrium point is stable because the arrow in interval (515584, ∞) is grows down and it show that the population of fish is decreased. However, the arrow in interval (264919, 515584) grows up and show that the population of fish is increased. The lower equilibrium point is unstable because the solution near the point is repelled.

Table 3 shows that the interval of equilibrium points whether the equilibrium is stable or unstable points.

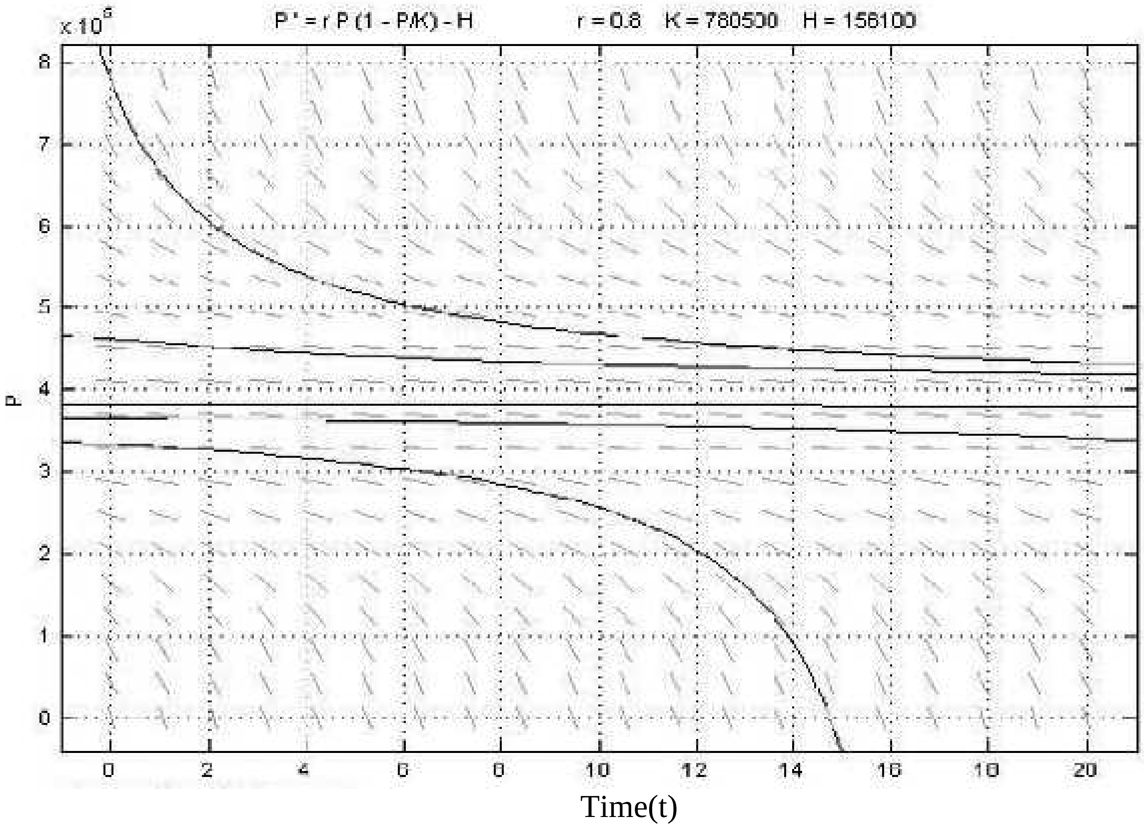


Figure 3.4.3: Harvesting = 156 100

Table 2. Interval of equilibrium points for harvesting = 156100

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 389482)$	Minus	Decreasing	Points Down
$(389482, \infty)$	Minus	Decreasing	Points Down

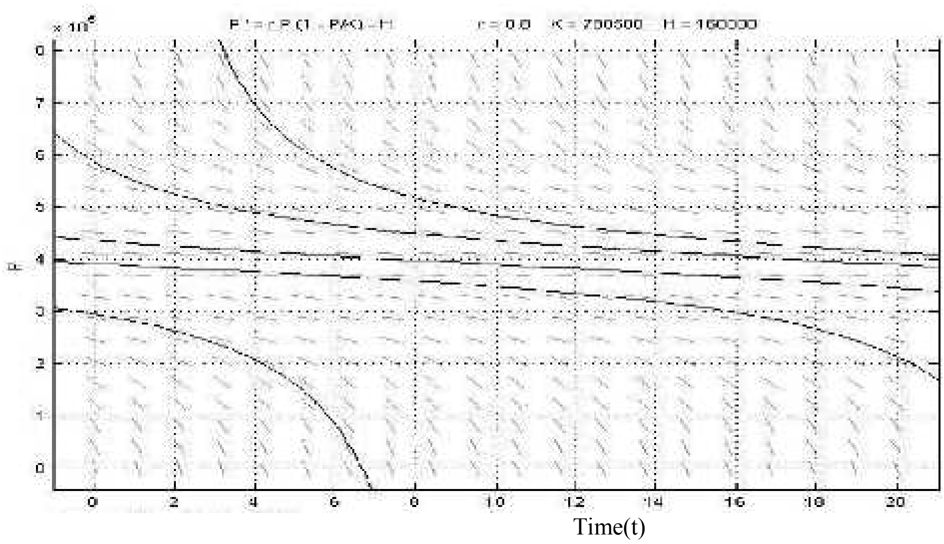
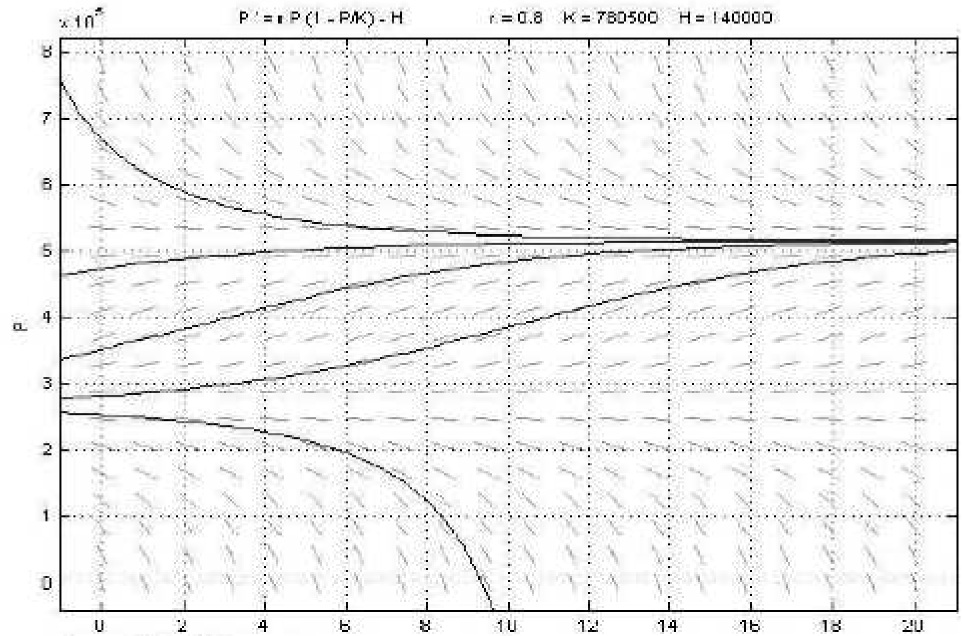


Figure 3.4.4: Harvesting = 160000 (For H > 156100)

Table 3. Interval of equilibrium points for harvesting = 160 000 (For $H > 156100$)

Interval	Sign of $f(P)$	$P(t)$	Arrow
(–∞, 264919)	Minus	Decreasing	Points Down
(264919, 515584)	Plus	Increasing	Points Up
(515584, ∞	Minus	Decreasing	Points Down



Time(t)

Figure 3.4.5: Harvesting = 160000 (For $H < 156100$)

Table 4. Results from constant harvesting strategies

$H = 156100$	$H < 156100$	$H > 156100$
One equilibrium point $P_0 = 389482$	Two equilibrium points $P_0 = 515584$ $P_0 = 264919$	No points of equilibrium exist
Equilibrium point give the initial population.	Upper equilibrium point that gives the initial population is stable. Otherwise the, the lower equilibrium point gives the unstable initial population.	All the considered initial population values will lead to extinction.

3.5 Prey and predator

The Lotka–Volterra equations, also known as the predator–prey equations, are a pair of first-order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey. They evolve in time according to the pair of equations:

$$dx/dt = \alpha x - \beta xy = x(\alpha - \beta y)$$

$$dy/dt = -\gamma y + \delta xy = y(-\gamma + \delta x) \qquad (5.1)$$

where,

- x is the number of prey (for example, rabbits);
- y is the number of some predator (for example, foxes);
- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the growth rates of the two populations over time;
- t represents time; and
- the constants α, β, γ and δ are all positive where α and γ are the growth rate constants and β and δ are measures of the effect of their interactions the two species
- when predators and prey interact, their encounters are proportional to the product of their populations, and each encounter tends to promote the growth of the predator and inhibit the growth of the pre

The above model is the simplest of the predator-prey models.

The Lotka–Volterra predator–prey model was initially proposed by Alfred J. Lotka "in the theory of autocatalytic chemical reactions" in 1910.The Lotka–Volterra system of equations is a more general framework that can model the dynamics of ecological systems with predator-prey interactions, competition, disease, and mutualism.

Physical meanings of the equations

The Lotka–Volterra model makes a number of assumptions about the environment and evolution of the predator and prey populations:

1. The prey population finds ample food at all times.
2. The food supply of the predator population depends entirely on the prey populations.
3. The rate of change of population is proportional to its size.
4. During the process, the environment does not change in favour of one species and the genetic adaptation is sufficiently slow

In the absence of the predator, the prey grows at a rate proportional to the current population; thus:

$$dx/dt = \alpha x, \quad \alpha>0 \quad \text{when } y=0$$

In the absence of the prey, the predator dies out, thus:

$$dy/dt = -\gamma y, \quad \gamma>0 \text{ when } x=0$$

when predators and prey interact, their encounters are proportional to the product of their populations, and each encounter tends to promote
the growth of the predator and inhibit the growth of the prey. Therefore, the predator population will increase by δxy. As a consequence of these assumptions , the Lotka-Volterra model is created:

$$dx/dt = \alpha x - \beta xy = x(\alpha-\beta y)$$

$$dy/dt = -\gamma y + \delta xy = y(-\gamma + \delta x)$$

The constants $\alpha, \beta, \gamma,$ *and* δ are all positive where α and γ are the growth rate constants and β and δ are measures of the effect of their interactions.

Dynamics of the system

In the model system, the predators thrive when there are plentiful prey but, ultimately, outstrip their food supply and decline. As the predator population is low the prey population will increase again. These dynamics continue in a cycle of growth and decline.

Population equilibrium

Population equilibrium occurs in the model when neither of the population levels is changing, i.e. when both of the derivatives are equal to 0.

$$x(\alpha - \beta y) - 0$$
$$-y(\gamma - \delta x) = 0$$

When solved for *x* and *y* the above system of equations yields

$$\{y = 0, x = 0\}$$ and
$$\left\{y = \frac{\alpha}{\beta}, x = \frac{\gamma}{\delta}\right\}.$$

Hence, there are two equilibria.

The first solution effectively represents the extinction of both species. If both populations are at 0, then they will continue to be so indefinitely. The second solution represents a fixed point at which both populations sustain their current, non-zero numbers, and, in the simplified model, do so indefinitely. The levels of population at which this equilibrium is achieved depend on the chosen values of the parameters, α, β, γ, and δ.

Stability of the fixed points

The stability of the fixed point at the origin can be determined by performing a linearization using partial derivatives, while the other fixed point requires a slightly more sophisticated method.

If $f = \frac{dx}{dt} = \alpha x - \beta xy$ and $g = \frac{dy}{dt} = \delta x - \gamma y$

The Jacobian matrix $J(x, y) = \begin{bmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{bmatrix}$

$$J(x, y) = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}.$$

The Jacobian matrix of the predator-prey model is

$$J(x, y) = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}.$$

First fixed point (equilibrium point)

When evaluated at the steady state of (0, 0) the Jacobian matrix J becomes

$$J(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = \alpha, \quad \lambda_2 = -\gamma.$$

In the model *α* and *γ* are always greater than zero, and as such the sign of the eigenvalues above will always differ. Hence the fixed point at the origin is a saddle point.

The stability of this fixed point is of importance. If it were stable, non-zero populations might be attracted towards it, and as such the dynamics of the system might lead towards the extinction of both species for many cases of initial population levels. However, as the fixed point at the origin is a saddle point, and hence unstable, we find that the extinction of both species is difficult in the model. If the predators are eradicated, the prey population grows without bound in this simple model).

Second fixed point (equilibrium point)

Evaluating *J* at the second fixed point we get

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = i\sqrt{\alpha\gamma}, \quad \lambda_2 = -i\sqrt{\alpha\gamma}.$

As the eigenvalues are both purely imaginary, this fixed point is not hyperbolic, so no conclusions can be drawn from the linear analysis. However, the system admits a constant of motion.

To find the equation of the curves traced out by trajectories. t must be eliminated

Using the chain rule, if $y(t) = y(x(t))$ then we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} ,$$

from which it follows that when $\frac{dx}{dt} \neq 0$ we have

$$\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}}$$

$$\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{-y(\gamma-\delta x)}{x(\alpha-\beta y)}$$

$$\left(\frac{\alpha-\beta y}{-y}\right)dy = \left(\frac{(\gamma-\delta x)}{x}\right)d$$

$$\left(\frac{\alpha}{-y}+\beta\right)dy = \left(\frac{\gamma}{x}-\delta\right)dx$$

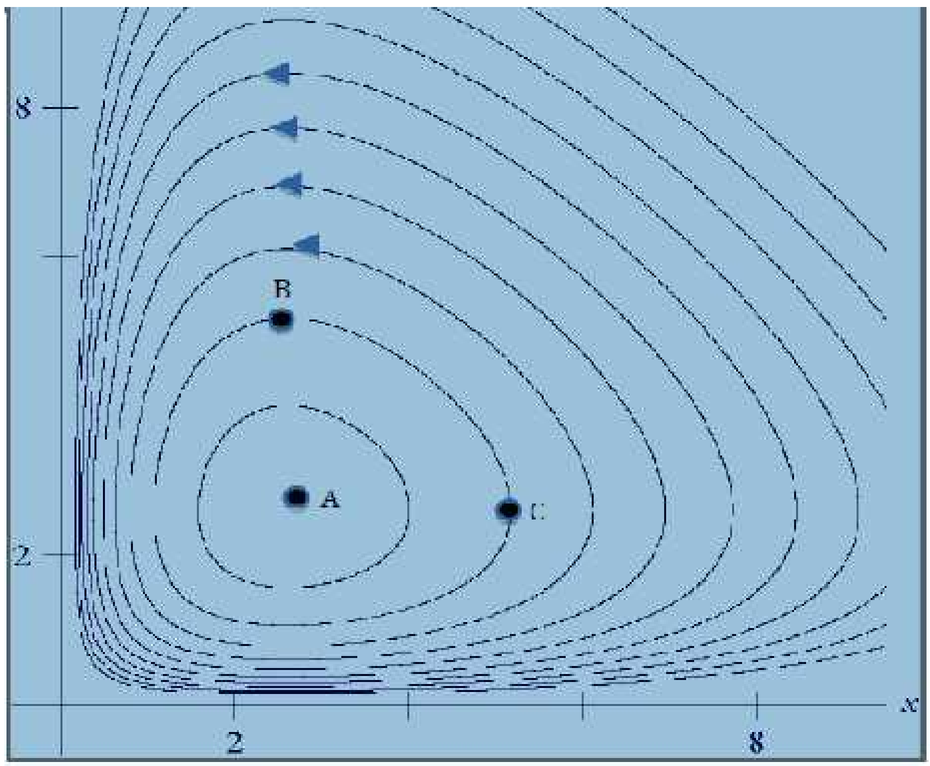
Integrating both sides gives (since *x* and *y* are positive

$$-\alpha \ln y + \beta y = \gamma\, lnx - \delta x + c'$$

$C(x,y) = \alpha\, lny + \gamma\, lnx - \delta x - \beta y$ is constant on trajectories. With some additional effort one can show that C(x, y)= constant define a simple closed curve. A numerical method should generate such a simple closed curve, and hence the trajectories form a collection of periodic orbits. Consequently, the levels of the predator and prey populations cycle, and oscillate around this fixed point .The constants can be changed and varied for different trajectories. A sample plot of the phase portrait of the Lotka-Volterra Model representing several different trajectories shown below in Figure (3.5.1)

Y

Predator



Prey

Figure(3.5.1) : Sample Lotka-Volterra Phase Portrait

In this plot, the x-axis represents population volume of the prey species, and the y axis represents the population volume of the predator species. Point A on the phase portrait represents an equilibrium point. In order to find equilibrium points of the system, each equation of the system must be set to equal zero, and all points that satisfy that condition are equilibrium points. Almost all such simple predator-prey models will have two equilibrium point somewhere in the first quadrant of the Cartesian plane, and the origin. Through analysis of the general solution of the linear system, it is discovered that the origin is a saddle point, with a shape defined by two linear trajectories known as the coordinate axes. Point A is a much different equilibrium point however ; By analyzing this equilibrium point using Jacobian Matrix analysis, the general solution shows this critical point of the linear system as a center, which is a stable (not necessarily asymptotically stable) critical point surrounded by infinitely many cyclic trajectories. Two points B and C were included in order to discuss phase portrait function. The purpose of this phase portrait is to show the cyclic fluctuations of the predator and prey species with respect to each other without showing the change in time. Let C=(5,3) and let B=(3,5) (not perfectly to scale as shown). These two points are on the same trajectory of the system, and with the direction of the trajectory, point C advances to point B as time goes on. One must understand that each point does not just represent arbitrary numbers, but each point represents the state of predator and prey population. Therefore, point C represents the state of the predation relationship at an arbitrary time, and as the time progresses the populations of each species shifts eventually to the state represented by point B. Given more time, eventually the population would return to the state of point C, which would complete one periodic oscillation. Find a different species community with the same constants (birth rate, death rate, kill rate), but different population volumes, and that predation relationship will be located on a different trajectory of the same phase portrait.

Another important plot stemming from the Lotka-Volterra model is the predator-prey cycle chart, representing periodic activity in the population fluctuation. This diagram is generated by plotting the y-t and x-t curves on the same plot, showing the fluctuation of predator and prey populations with respect to time on the same chart, therefore important characteristics of time can be analyzed. Figure 2 below depicts a sample predator-prey cycle chart.

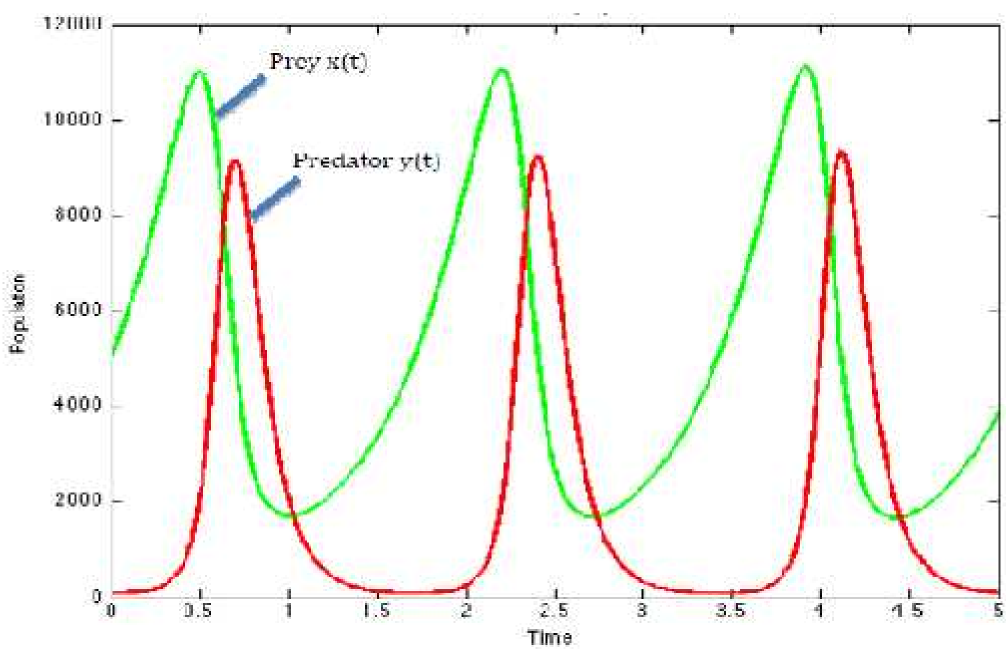


Figure (3.5.2) : Predator-Prey cycle chart.

It is seen that as time progresses (in years), predator and prey populations clearly fluctuate at cyclic interval. Notice that as the prey population peaks, predator population begins to rise rapidly, yet as the predator population rises, the prey population falls rapidly. Then follows a longer period as the prey population must slowly repopulate and the predator population falls drastically. This cycle repeats itself over and over in reality, which is why biological mathematics can attempt to recreate the pattern mathematically.

One obvious shortcoming of the basic predator-prey system is that the population of the prey species would grow unbounded, exponentially, in the absence of predators. There is an easy solution to this unrealistic behavior.We'll just replace the exponential growth term in the first equation of equation (6.1) by the two-term logistic growth expression and we get

$$dx/dt = \alpha x - rx^2 - \beta xy = x(\alpha - r) - \beta xy$$

$$dy/dt = -\gamma y + \delta xy = y(-\gamma + \delta x) \qquad (5.2)$$

The constants $\alpha, \beta, \gamma,\ \delta$ and r are all positive

Therefore, in the absence of predators, the first equation becomes the logistic equation. The prey population would instead stabilize at the environmental carrying capacity given by the logistic equation.

Analyzing the Prey- Predator Equation:

$$dx/dt = \alpha x - \beta xy = x(\alpha - \beta y)$$

$$dy/dt = -\gamma y + \delta xy = y(-\gamma + \delta x)$$

Data analysis is based on the following system:

$$dx/dt = x(0.5-0.02y)$$

$$dy/dt = y(-0.9+0.03x)$$

This system was created based on using the constants $\alpha = 0.5$, $\beta=0.02$, $\gamma=0.9$ and $\delta=0.03$.

These constants come from determining the best possible parameters of the snowshoehare and Canadian Lynx population data shown in the table below.

Year	Hares(X 1000) (predator)	Lynax(X 1000) (Prey}	Year	Hares(X 1000) (predator)	Lynax(X 1000) (Prey}
1900	30	4	1911	40.3	8
1901	47.2	6.1	1912	57	12.3
1902	70.2	9.8	1913	76.6	19.5
1903	77.4	35.2	1914	52.3	45.7
1904	36.3	59.4	1915	19.5	51.1
1905	20.6	41.7	1916	11.2	29.7
1906	18.1	19	1917	7.6	15.8
1907	21.4	13	1918	14.6	9.7
1908	22	8.3	1919	16.2	10.1
1909	25.4	9.1	1920	24.7	8.6
1910	27.1	7.4			

This data was inputted into Microsoft Excel and the Solver application was used in order to discover the best possible fit parameters to the desired Lotka-Volterra model.

Discovering Critical Values

The first step in analyzing the system is discovering the critical points by setting each of the two equations in the system to equal zero and determining analytically the satisfying points. The two critical points of this equation are (0, 0) and (30, 25) (notice this result in the origin and one point in the first quadrant as previously stated).

Computer Simulations: Shown below is a MATLAB rendered direction field of the system in Figure 3.5.3 . Notice that the point (0, 0) is a saddle point and the critical point (30, 25) is a center point, or equilibrium point.

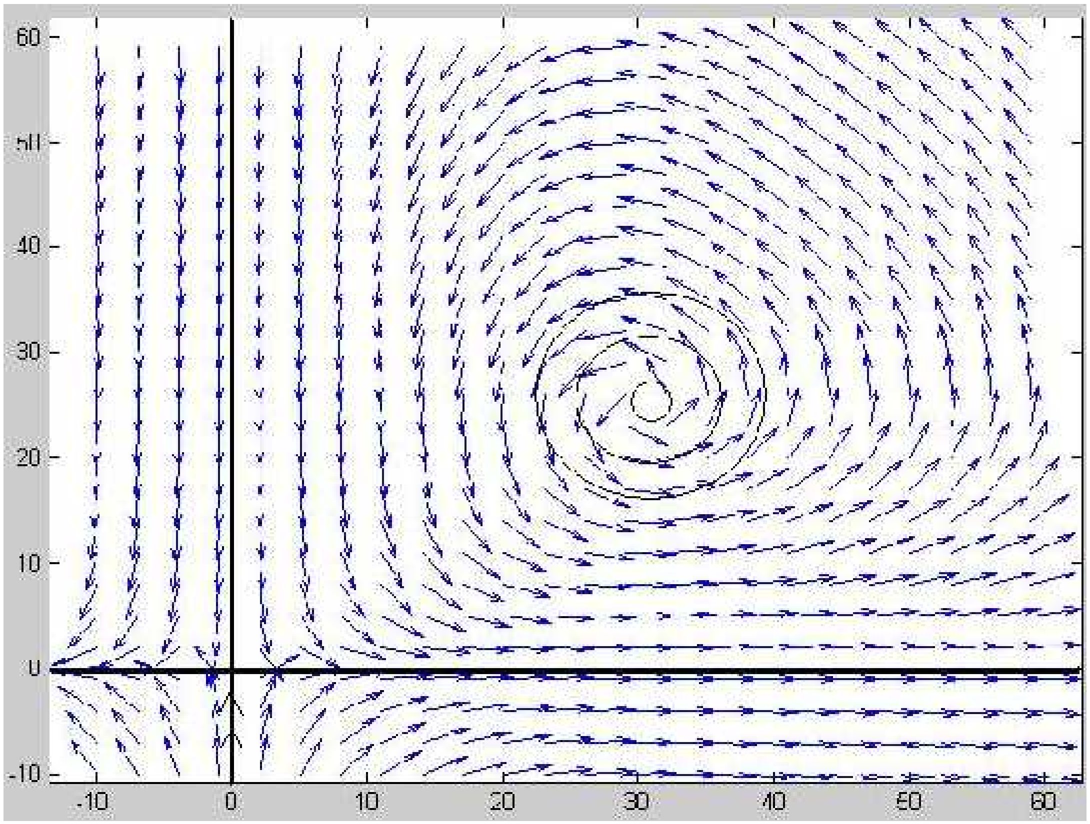


Figure 3.5.3 : MATLAB rendered direction field of the system, notice critical point at (30,25)

By examining the corresponding linear systems to the solutions near each critical point we can determine local behavior of each point. The general solution of the linear system corresponding to the origin shows that, based on local behavior, this point is a saddle point. However, the critical point at (30, 25) is much more difficult to analytically determine the behavior. Using Jacobian Matrix analysis, the corresponding linear system can be obtained, and the eigenvalues of this system are imaginary. Imaginary eigenvalues equivocates to the critical point being a center. The nonlinear systems of each of these critical points share the behavior of their linear counterparts. This direction field shows the general direction of the entire field at large. As shown, in figure 3.5.3 the infinitely many trajectories that exist outside of the critical points take on a similar directional pattern. Since the point (30, 25) is a center, all trajectories surrounding this point are known as cyclic variations, or periodic oscillations. While not accounting for time, the main purpose of this visual is to notice the ever-changing pattern of the two different populations with respect to one another.

The next critical graph to examine is the Predator-Prey Cycle Chart shown in Figure 3.5.4 below. This graph depicts population (in thousands) versus time (in years) of both species on the same graph. This is quite useful in order to visualize the population fluctuations of each species also with respect to time. The average time of the periodic oscillation can be determined graphically in this way, and general population variation characteristics can be determined. Notice the reality represented by the graph, as the prey population peaks, the predator population begins to rapidly increase. Just as the predator population peaks, the prey population begins to drop rapidly. The curve for the Lynx follows behind the Hare curve in the same shape and pattern.

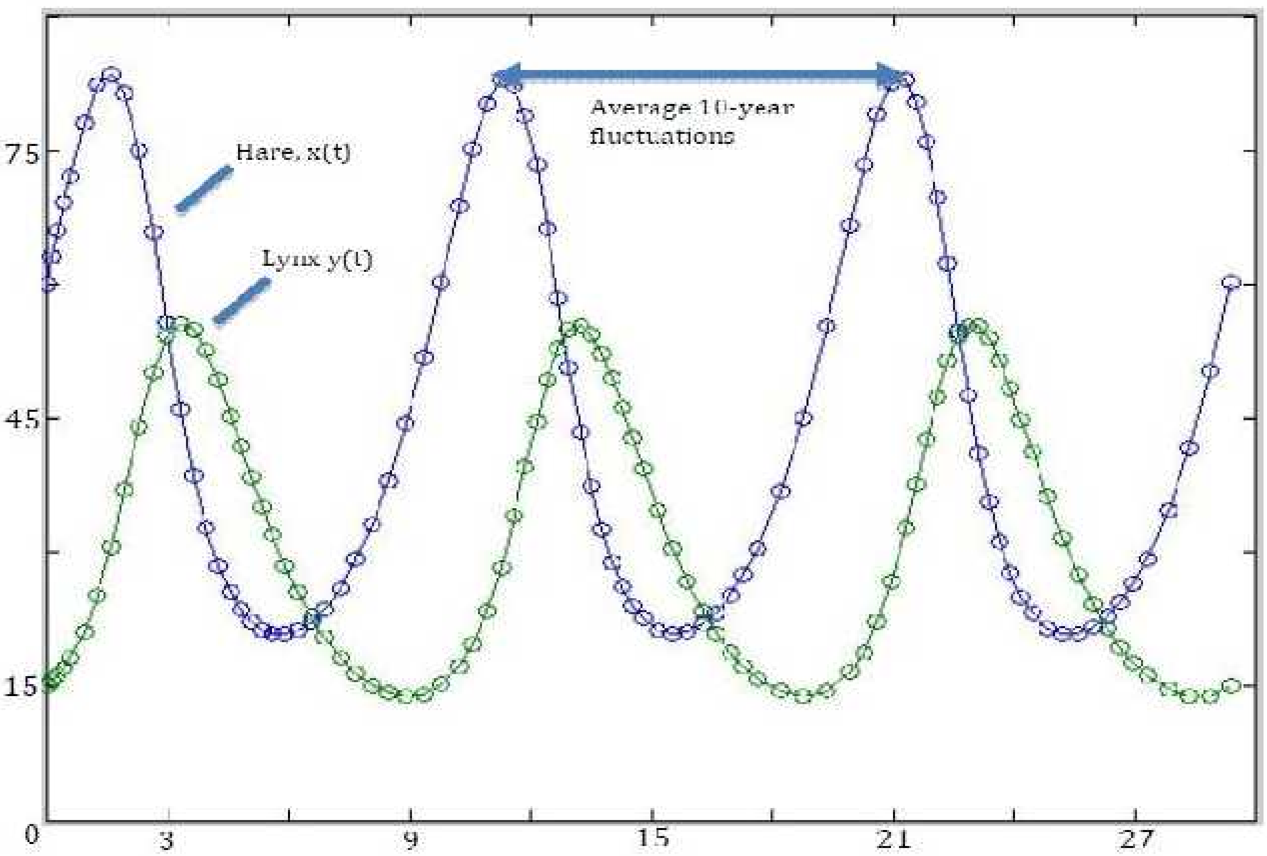


Figure 3.5.4 : a MATLAB rendered Predator-Prey Cycle Chart of the system.

One of the most important values found in this graph is the average periodic oscillation. By analyzing the same point in sequential phases and finding the time in between them, the periodic oscillation can be determined. Often peaks of the curves are used for this. As shown in Figure (3.5.4) above, the average periodic oscillation for the Snowshoe Hare and Canadian Lynx is a ten year fluctuation.

The Lotka-Volterra Predator-Prey Model is a rudimentary model of the complex ecology of this world. It assumes no factor influence like disease, changing conditions, pollution, and so on. However, the model can be expanded to include other variables, and we have Lotka-Volterra Competition Model, which models two competing species and the resources that they need to survive. We can polish the equations by adding more variables and get a better picture of the ecology.

3.6 A Falling Object with Air Resistance

According to Newton's first law of motion, a body will either remain at rest or will continue to move with a constant velocity unless acted upon by an external force. This implies that if the net force (resultant force) acting on a body is zero then acceleration of the body is zero. According to Newton's second law of motion, the net force on a body is proportional to its acceleration, provided the net force acting on the body is not zero. That is if F≠0 is net force, m is the mass of the body and a is the acceleration then F=ma.

In most falling-body problems we neglected air resistance . In this section the air resistance on the falling object is assumed to be proportional to its velocity *v*. If *g* is the gravitational constant, the downward force *F* on a falling object of mass *m* is given by the difference *mg* - *kv* . But by Newton's Second Law of Motion, F = ma = $m\,\frac{dv}{dt}$

which yields the following first order differential equation

$$m\,\frac{dv}{dt} = mg - kv \tag{6.1}$$

$$\frac{dv}{dt} + \frac{k}{m}\,v = g \tag{6.2}$$

Example 3.6.1 : An object of mass *m* is dropped from a hovering helicopter. Find its velocity as a function of time *t*, assuming that the air resistance is proportional to the velocity of the object.

Solution : The velocity *v* satisfies equation (6.2)

$$\frac{dv}{dt} + \frac{kv}{m} = g$$

where *g* is the gravitational constant and *k* is the constant of proportionality.

Letting $b = k/m$ you can *separate variables* to obtain

$$dv = (\; g - bv\,)$$

$$\int \frac{dv}{g-bv} = \int dt$$

$$\Rightarrow\; -\frac{1}{b}\ln|g - bv\,| = t + g$$

$$\ln|g - bv\,| = -bt - bc_1$$

$$g - bv = c\,e^{-bt}$$

Because the object was dropped, v= 0 when t = 0, then g =0 and it follows that

$$-bv = -g + ge^{-bt}$$

$$\Rightarrow\; v = \frac{g - ge^{-bt}}{b} = \frac{mg}{k}\left(1 - e^{-kt/m}\right) \tag{6.3}$$

Sky diving :

Once the sky diver jumps from an airplane, there are two forces that determine his motion : The pull of the earth's gravity acting down ward and the opposing force of air resistance acting upward. At high speeds, the strength of the air resistance force (the drag force) is proportional to the square of velocity , so the upward resistance force due to the air resistance can be expressed as $F_{air\ resistance} = kv^2$, where v is the speed with which the sky diver descends and k is the proportionality constant determined by such factors as the diver's cross-sectional area and the viscosity of the air. Once the parachute opens, the decent speed decreases greatly, and the strength of the air resistance force is given by kv.

By Newton's Second Law F_{net} = ma

Since the acceleration is the time derivative of the velocity, this law can be expressed in the form

$$F_{net} = m\frac{dv}{dt} \qquad (6.4)$$

In the case of sky diver initially falling without a parachute, the drag force is

$F_{drag} = kv^2$, and the equation of motion (8.4) becomes

$$mg - kv^2 = m\frac{dv}{dt}$$

Or more simply, $\frac{dv}{dt} = g - bv^2$

Where $b = k/m$. The letter g denotes the value of gravitational acceleration, and mg is the force due to gravity acting on the mass m(that is , mg is 'its weight). Near the surface of the earth, g is approximately 9.8 $^{m}/_{s^2}$. Once the sky diver's descend speed reaches $v = \sqrt{mg/k}$ $^{m}/_{sec}$, The preceding equation says $dv/dt = 0$; that is, v stays constant. This occurs when the speed is great enough for the force of air resistance to balance the weight of the sky diver, the net force and the acceleration drop to zero. This constant descent velocity is known as the *terminal velocity*. For the sky diver falling in the spread-eagle position without parachute the value of the proportionality constant k in the drag equation $F_{drag} = kv^2$ is approximately $\frac{1}{4}\ kg/_{m}$. Therefore, if the sky diver has a total mass of 70 kg (which corresponds to a weight of about 150 pounds), his terminal velocity is

$$v_{\substack{terminal \\ (no\ parachute)}} = \sqrt{\frac{mg}{k}} = \sqrt{\frac{(70)(9.8)}{\frac{1}{4}}} = 52\ m/h$$

Or approximately 120 miles per hour.

Once the parachute opens, the air resistance force becomes $F_{air\ resistance} = Kv,$ and the equation of motion (8.4) becomes

$$mg - Kv = m\frac{dv}{dt}$$

Or more simply ,

$$\frac{dv}{dt} = g - Bv \qquad (6.5)$$

Where B =$k/m\ \ ^{1}/_{sec}$. Once the parachutist descent speed slows to $v = g/B = mg/K$, the preceding equation says $\frac{dv}{dt}$ = 0, that is v stays constant .This occurs when the speed is low enough for the weight of the sky diver to balance the force of air resistance, the net force and the acceleration reaches zero, again this constant descent velocity is known as the terminal velocity. For the sky diver falling with a parachute, the value of the proportionality constant K in the equation $F_{resistant} = kv$ is approximately 110 kg/s .

Therefore, if the sky diver has total mass of 70 kg, the terminal velocity with parachute open is only

$$v_{\substack{terminal \\ (with\ parachute)}} = \frac{mg}{K} = \frac{(70)(90)}{110} = 6.2\ m/s$$

Which is about 14 miles per hour. Since it is safer to land on the ground while falling at a rate of 14 miles per hour rather than at 120 miles per hour, sky diver use parachutes.

Example 3.6.2 : After a free falling sky diver of mass m reaches a constant velocity v_1, his parachute opens, and the resulting air resistance force has strength kv . Derive an equation for the speed of the sky diver at t seconds after the parachute opens.

Solution: Once the parachute opens, the equation of motion is

$$\frac{dv}{dt} = g - Bv$$

where B= K/M . The parameter that will arise from the solution of this first order differential equation will be determined by the initial condition v(0) = v_1 (since the sky divers velocity is v_1 at the moment of the parachute opens, and the "Clock " is reset to t=0 at this instant). This differential separable equation is solved as follows:

$$\int \frac{dv}{g-Bv} = \int dt$$

$$\frac{-1}{B}\, ln(g - Bv) = t + c'$$

$$ln(g - Bv) = -Bt + c'$$

$$g - Bv = \mathrm{c}e^{-Bt}$$

Now, Since V(0) =v_1 $\Rightarrow g - Bv_1 = c,$ The desired equation for the sky diver's speed at t seconds after the parachute opens is

$$g - Bv = (\,g - BV)e^{-Bt}$$

$$Bv = g - (\,g - BV)e^{-Bt}$$

$$v = \frac{g}{B}\left|1 - \left(1 - \frac{B}{g}\,v_1\right)e^{-Bt}\right|$$

$$v = \frac{mg}{K}\left|1 + \left(\frac{K}{mg}v_1 - 1\right)e^{-(K/m)t}\right|$$

Note that as time passes that is, as t increases , the term $e^{-(K/m)t}$ goes to zero, so the parachutist's speed v slows to mg/k , which is the terminal speed with the parachute open.

SUMMARY

This seminar attempted to discus the application of first order differential equation by using modeling phenomena of real world problems. Some of the models included are Newton's cooling or warming law, Population dynamics, Prey and predator , Harvesting of renewable natural resources and a falling object with air resistance .

Mathematically it is possible to represent the population variations of prey and predator relationship to a certain extent of accuracy by using the Lotka-Volterra model which is described by systems of linear differential equation . This system of linear first order differential equations can be used to interpret analytically and graphically the cyclic fluctuations of the prey and predator populations and also mathematical model have been used widely to estimate the population dynamics of animals as well as the human population . From this discussion we get some idea how differential equations are closely associated with physical applications and also how the law of nature in different fields of science is formulated in terms of differential equations

As a conclusion many fundamental problems in biological, physical sciences and engineering are described by differential equations. It is believed that many unsolved problems of future technologies will be solved using differential equations. On the other hand, physical problems motivate the development of applied mathematics, and this is especially true for differential equations that help to solve real world problems in the field. Thus, making the study on applications of differential equation and their solutions essential with this regard.

REFERENCES

A.C. King, J. Billingham and S.R.Otto, 2003, *Differential equations (Linear, Non linear, Ordinary , partial)*, Cambridge University Press

AH Siddiqi and P. Manchanda, 2006, *A First course in differential equation with application*, Rajiv Berifor Macmillarie India Ltd

Dawkins, Paul, 2011, *Differential equations*, Lamar University

Fowler, Andrew, 2005, *Techniques of Applied Mathematics*, University of Oxford

George F.simmons , *Differential Equation with Applications and Historical notes, 2nd* edition, Tata Mcgraw Hill education private limited

Kent, Nagle R. and Edward B. Saff, 1996, *Fundamentals of Differential Equations and boundary value problems.2nd* edition, Addison-Weseley

Kreyzing, Erwin , 1993, *Advanced Engineering Mathematics*. 7th Edition, Wayne

Markowich, PeterA, 2006, *Applied Partial Differential Equations*, Springer

Marthinsen, K. Engo, A,1998, *Modeling and solution of some mechanical problems on Lie groups*, Multibody System Dynamics 2, Newyork: springer-verlage

Maurrice D.Weir, Joel Hass and Frank R.Giordano, Thomas' calculus, 11th edition, Doring Kindersely India Pvt.Ltd

Tao.Ram, *Application of first differential equation in Mechanical engineering analysis*, San. Jose state University