www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

SCUDEM 2017 Author Commentary - Overview

Kelly Black
Department of Mathematics
University of Georgia

OVERVIEW

For SCUDEM 2017, the Inaugural Student Competition Using Differential Equation Modeling, held at Mount Saint Mary College, Newburgh NY USA, on Saturday, 14 October 2017, undergraduate teams of three students each were given three different modeling problems and asked to choose one to work on. The problems for selection were made available to student teams on the morning of Monday, 9 October 2017, and teams worked during the week to prepare an Executive Summary and a 10 minute Presentation to be offered for judging on Competition Saturday, 14 October 2017. The problems were complex and came from varied contexts. It is not possible to provide a complete, unique answer in a short amount of time. According to self-reported data, the average individual time spent on the competition from first seeing the problems on Monday, 9 October 2017, until Competition Saturday morning was 14.5 hours.

The teams worked diligently to provide some insight into their selected problem. A brief overview of the teams' efforts is given here. Each of the three problems is examined separately. Additionally a broad overview is given that features common observations of the entries from all teams. We present the statements of each of the problems in Appendix 1.

On the morning of Competition Saturday the teams were issued a supplemental issue for their chosen problem and the students were asked to describe how they would alter their model to address this new issue. They were not asked to redo the model, but to describe what changes they might make and the resulting change in their model outcomes. We present the statements of each of the supplemental problem issues in Appendix 2.

Problem A - Going Viral

See Appendix 1 for complete text of all three problems.

The context for the first problem is an advertising firm. The members from the firm would like to create a social media message that will go viral. The central question is to determine the resources required to promote the message. A number of other questions are included to help the firm determine how to manage the message once they begin to promote it.

A supplemental issue was also presented to the students on the morning of the Competition and the students were asked to describe how they would extend their model to adapt to the new criteria. They were not asked to redo the model, but to describe what changes would be required. For this supplemental problem, the students were asked to determine how their model could be adapted to allow the advertising firm to quickly decide if the message was meeting its target schedule. If the target schedule is not being met then the team is asked how the model could help determine how to bring the status of the message back on schedule.

With respect to the modeling efforts, many of the teams started with either an SIR or an SEIR model. One of the teams adapted the SEIR model to fit this particular context. The adaptation of the SEIR model was an interesting approach and showed some interesting modeling as the teams transformed an established model for use in a different context.

One of the teams made use of a Logistic model. This approach is a little problematic. The behavior of the model is fixed and only includes one variable. In this case the way a message is shared can take many different forms, and the use of a Logistic model will make it difficult to determine what approach can yield the particular desired behavior.

In terms of the modeling process, one of the difficulties in this problem is to carefully define the problem and describe the desired behavior. In particular, the question statement does not have a formal statement of what it means for a message to go viral. In this case adapting the model required the teams to define what the different terms in the model represented, and it was difficult to determine the units used in the model. The teams were able to address this difficult aspect of the problem with varying degrees of effectiveness. The teams that were able to clearly define these details made an immediate positive impression.

Another important thing to note is the way the teams analyzed their model. For example, one of the teams discussed the equilibrium of their model. This is an important aspect of any model. Teams are not expected to provide an elaborate model with a complete solution. It is important, though, to be able to develop a basic model that can capture some of the phenomena of interest. The key aspect of modeling is to start with a simple model and determine how to adapt and change the model to better reflect the phenomena of interest.

Teams that can provide insight and analysis of their model are taking part in one of the most important aspects of the process of modeling. To be able to explore and assess the general behavior of a model is a necessary step in making changes and improvements to a model.

Problem B - Drug Interactions

See Appendix 1 for complete text of all three problems.

The context of the second problem is the interaction of two drugs. The complication is that one of the drugs is a diuretic which is designed to stimulate the kidneys to remove water from a patient. This impacts the schedule used to administer other drugs, as it can hasten the removal of the other drugs from a patient's circulatory system.

A supplemental issue was also presented to the students on the morning of the Competition and the students were asked to describe how they would alter their model to accommodate the different ways that a medication can be administered to a patient. They were not asked to redo the model, but to describe what changes they might make and the resulting change in their model outcomes. For this supplemental problem they were asked to decide what changes were necessary if a patient is asked to alter her water intake, such as by a continuous method, such as an intravenous solution, or a discrete method, such as drinking more fluids than normal.

One of the surprises in reading the responses to this question is that many teams started with a complicated model. My original expectation was for students to begin with a simpler linear model and then build on that model with small changes as time permitted. The advantage of the linear model is that the resulting analysis can proceed using methods found in many introductory textbooks.

Of the entries that addressed this question, one team was more careful about providing details about their approach. Notably, the team explicitly defined and derived the variables of interest. In this case the units were easier to understand, but it was still important to note the difference between conserved quantities such as the amount of drug versus quantities not conserved such as the concentration of the drug.

Another notable aspect of the reports is that one of the teams provided some insight into the sensitivity of their model looking at how the solutions change with different situations. The notion of sensitivity is a crucial aspect of the modeling process. To be able to identify which parameters or terms have the biggest impact in the solution of a system is a vital step necessary to determine which parts of a model have the biggest impact in the large scale behavior of the system.

Problem C - Game Play

See Appendix 1 for complete text of all three problems.

The context for the third problem is to determine a way to generate paths for objects moving in space for a science fiction movie. Surprisingly, only one team examined this question. This was not expected because I assumed this problem would have some appeal with respect to the topic, and the approach to the problem would be easier to see after the first reading of the question.

A supplemental issue was also presented to the students on the morning of the Competition and the students were asked to describe how they would alter their model to accommodate the different ways that a medication can be administered to a patient. They were not asked to redo the model, but to describe what changes they might make and the resulting change in their model outcomes. For this supplemental problem the students were prompted to engage in a sensitivity analysis of their question. The team went back and examined their model to determine which aspects of the model as well as their assumptions had the biggest impact on the game.

The team that explored this problem presented a substantial model, and the team presented a number of numerical approximations of their model. It was not clear how the results were generated, though. Without the details of the approximation it was difficult to get a feel for how robust the results are.

The team did an excellent job of providing an overview of the problem. This is something that most other teams did not do. The team's efforts also stand out for the way the team discussed the different parts of the problem and provided details about each of the individual terms in their model. The team then brought the different pieces together in a coherent way using Newton's Second Law.

COMMON OBSERVATIONS

Finally, there were a number of common things that were found across all of the teams' entries. These include some basic modeling issues as well as writing issues. The teams' writing is just as important as the mathematics, and the two things cannot be separated. A team's efforts are only as good as their ability to share, exchange, and communicate the ideas they develop.

Introducing The Problem

The majority of the reports did not include a comprehensive description of the problem. A report should stand on its own, and the team members should not assume that the reader has seen the original problem statement. A technical report should start with an introduction to the problem and provide a broad overview of the team's approach. A report should also include a conclusion that succinctly brings everything together.

Approximation Is Complicated

Many of the teams presented approximations to their models, but offered little direction as to how the approximations were generated. The use of approximations is expected, especially when the event includes a difficult time constraint. Constructing an approximation, however, can be a non-trivial part of a team's efforts, and it is important to let the reader know how the approximation was constructed.

Related to this part of the teams' reports, the teams chose values for parameters in the approximation, but many did not make it clear how the values were chosen. The behavior of a system can vary widely based on the values of some constants.

Choosing values for the parameters is a critical part of the process, and the reader should be informed how this step was undertaken.

Units and Small Details

One of the difficulties associated with a question that does not have a well-defined central question is that there are many different ways to define the quantities of interest. It is important to explicitly define the quantities and units that a model is tracking. It is also important to note the difference between the variables of interest and the rates of changes of the variables.

The definition of the units is a small detail that helps the reader understand how the development of a model proceeds. Other details that are important to share are the basic physical principles employed and the various ways the changes in the variables are being tracked. Providing the motivation as to how the different phenomena manifest themselves in the mathematical model allows the reader to understand the motivation for the construction of the model and also helps the reader determine if the goals of the model and the implementation are consistent.

Missing Items

There were some other important parts of the modeling process itself that were missing in many of the teams' reports. The first thing that was missing was an explicit discussion of the teams' basic assumptions. One of the first steps in building a model is to define the basic assumptions. Likewise, the assumptions should be stated throughout the narrative of a team's report.

Another important thing that was missing in many reports was the use of both citations and references. Several of the teams provided a list of references used by the team, but none of the teams provided citations within their narrative. The lack of a consistent use of citations within the text made it extremely difficult to determine how the teams used their sources. It was not clear what parts of the model came from the team and what parts were found in a team's reading materials.

Stability and Sensitivity Analysis

The modeling process is a cycle in which a simpler model is examined and refined which leads to the adaptation and development of a new model based on the prior model. Before a model can be updated the general behavior of the model must first be determined. The model and the analysis of the model cannot be separated since they are both used to further refine and define the next version of the model.

Two kinds of analysis are used to determine the stability and sensitivity of the model. Stability is used to determine the long term behavior of a solution. Sensitivity is used to determine which parts of the model have the greatest influence with respect to the short term evolution of a solution to the equations used in the mathematical model.

To illustrate the kind of analysis that might occur, the second problem (the interaction of two drugs), is briefly examined. I assume that the first drug, the rate the diuretic changes in the system does not depend on the first drug, and the rate that it is removed from the blood stream depends on the amount present. So if $A(t)$ is the mass of the total amount of the diuretic in a person's blood stream, then

$$\frac{dA}{dt} = -kA, \tag{1}$$

where k is a positive constant.

Next, I assume that the second drug is eliminated at a rate that depends on both the amount of the diuretic and the amount of the second drug. The total mass of the second drug in the blood stream is denoted as $B(t)$, and if the effect is additive then

$$\frac{dB}{dt} = -lA - mB, \tag{2}$$

where l and m are positive constants.

The result is a linear system of differential equations,

$$\frac{d}{dt}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -k & 0 \\ -l & -m \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} \tag{3}$$

In this case an exact solution can be found, and the eigenvalues are $-k$ and $-m$. The solution decays, and the limiting rate of decay is whichever value is larger, $-k$ or $-m$.

The resulting solution to this system is problematic. The eigenvectors indicate that if $-k$ is greater than $-m$ then the second drug will be removed from the system more slowly than if the diuretic were not present. This indicates that a second model when developed should focus on the interaction between the two, and a nonlinear term should first be added to the second equation. It also indicates that with respect to the sensitivity of the model, whichever value, $-k$ or $-m$ is greater, then small changes in that variable will yield the greatest change in the overall system.

The resulting model described above is problematic, and it does not adequately capture the phenomena of interest. From the standpoint of mathematical modeling, though, it is a first step, and a formal analysis of the problem is given. Here, the motivations for the model are briefly stated, and an analysis of the model is provided. The groundwork has been provided to go back and make a change to the model. Once that change is made, the analysis begins again, and both the good and the bad aspects of the resulting model can be determined.

The goal is not to make a great model since there is not enough time. Instead, the goal is to demonstrate the process of applying basic principles, analyze the system, identify shortcomings, and then adapt the model. The simplest model that can capture the basic nature of the important phenomena can provide more insight into a problem then a complex model that is able to demonstrate the finest minutiae.

CONCLUSION

Teams of students were brought together as part of an event to promote and support the use of mathematical modeling with differential equations. The teams were given three different problems and were asked to explore the solution to the questions associated with the problems. The problems included a requirement of constructing a model to describe how a message goes viral, a question on how drugs interact within a patient's body, and a question about how to generate paths for objects moving in a scene from a movie, respectively.

The teams presented their results in two forms, a two page Executive Summary and a ten minute Presentation. The approaches used by the teams varied widely in their complexity. Most of the teams tried to develop sophisticated models with basic levels of analysis and approximation. It is important, though, to provide a wider view into the process of developing a mathematical model.

In particular, the process begins with a simpler model, and a more advanced model is slowly developed after a detailed analysis of the model. The teams taking part in the event provided varying degrees of analysis. Most teams were able to provide some analysis, but many teams placed a greater emphasis on the creation of more sophisticated models at the expense of limiting their ability to use analytic tools that could be used to determine the broad behavior of the solutions to the resulting models.

APPENDIX 1 - PROBLEM STATEMENTS

PROBLEM A - Going Viral

Advertisers strive to reach the greatest number of people using the least amount of resources. The prevalence of social media and easy access to digital media has resulted in the development of strategies to create advertisements that go "viral." Advertisers want a large number of people to share their message in ways that the number of people seeing the message grows rapidly in a short time.

This has been done in a variety of ways. One way is to target specific groups of people who are known to have broad audiences that can spread a message quickly. Another way is to make an advertisement available to a great number of social media outlets in a short time span. The hope is that the advertisement will be shared by enough people that the sharing itself will gain momentum and increase rapidly.

A good deal of research has been conducted to model the way information spreads. Many of these models focus on the structure and size of the network of people. In practice the networks of people are already established and can be vast and quite complicated. Is it possible to construct a simpler model that can describe the way the number of people are exposed to an idea and predict viral growth?

In particular, an advertising company would like to know the minimal amount of resources that can be spent and result in a rapid expansion of the number of people exposed to a message. For example, is there some minimal number of people who must be exposed to their message in a short time to cause the message to go viral? If an advertisement does not go viral quickly after release how likely is it that it will never go viral, or is it possible to add resources in a way that can promote an advertisement so that it does go viral if at first it does not appear to be successful?

Problem B - Drug Interactions

Patients undergoing medical treatment may be given a combination of drug therapies, and the information about the interaction between drugs can be limited[1]. One commonly prescribed class of drugs is diuretics which promote kidney function as a way of removing water and sodium from the circulatory system. The way that a diuretic can interact with other drugs can be quite complicated and depends on the type of diuretic.

We ask that you examine the simpler issues of the reduction or removal of a compound from within a patient's system. A drug can be reduced within a patients system in several different ways. For example, it can be metabolized and broken down within the liver, and a patient's kidneys can directly remove it from the bloodstream.

You are asked to provide an analysis that will give direct guidance for the administration of a drug. The patient will be given a diuretic to relieve symptoms associated with heart disease. At the same time a patient will be given another drug to help treat other symptoms. The goal is to maintain a consistent level of both drugs within a patient's circulatory system.

The issue is that the diuretic will promote a more rapid removal of both drugs from the patient's system, but at the same time the other drug must be maintained at levels that are effective and safe. The medical staff would like to know how to balance the administration of the two drugs. They need to know what schedule and what dosages are appropriate for a given situation.

1. Egger, S. S., J. Drewe, and R. G. Schlienger. 2003. Potential drug-drug interactions in the medication of medical patients at hospital discharge. *Eur. J. Clin. Pharmacol.* 58: 773-778. doi:10.1007/s00228-002-0557-z.

Problem C - Game Play

A new game is proposed for a hand-held device. The game needs to be relatively simple and the computational resources required must be kept small. The proposed game is a network based game, so two players can play at the same time on their separate devices.

The concept is that there is an animated ping pong ball that moves through an obstacle course and its motion is controlled in turn by the two players. The first player moves a paddle to hit the ball into the obstacle courses. When the ball reaches the other end of the obstacle course the second player must move a paddle to return the ball to the first player who then maneuvers the ping pong ball back to the second player on the other end of the obstacle course. The process repeats until a player is unable to return the ball, thus giving the player who misses one point. Much like table tennis when a player reaches 21 points and the margin in points is 2 or more the player with the higher points loses and a winner is declared. Service alternates and either player can score on service.

The ping pong ball must move in two dimensions, left- right and forward- backward. The ball is exposed to friction so it will slow down if no other forces act on it. The players can provide a boost by pushing the ball with a force that can be forward or backward and a force that can be left or right. The player must use the forces to move the ball through the obstacle course as quickly as possible while avoiding any obstacles.

You are asked to develop the system of equations that describe the motion of the ping pong ball. You should also provide recommendations to the software developers. They need to know the levels of force necessary to make the game interesting, yet still playable. They also need to know what limitations should be put in place so that the game will be challenging, but not too easy. For example, there should be a limit on the amount of propellant available to apply force to the ping pong ball. The developers want players to develop a strategy on how to use the limited propellant allowed. Since these decisions depend on the number and size of the obstacles you should also provide recommendations about the obstacle course and overall game play that will make the experience as enjoyable as possible.

APPENDIX 2 - SUPPLEMENTAL PROBLEM STATEMENTS

Problem A - Going Viral

An organization is considering using your model but wants a change. They want to also have a prediction as to how long it will take for the information they want to spread to hit the peak transfer rate. They want the model adapted so that if the methods in use will not result in a quick dissemination they can make changes to the approach to insure that the information is spread more rapidly. Determine what should be monitored, and how that additional information can be used to make changes to the recommendations.

Problem B - Drug Interactions

As part of the treatment a patient may be given additional fluids to counteract the loss of fluids due to the diuretic. Some patients will be given the additional fluids using an intravenous fluid therapy, and the fluids will be introduced at a relatively constant rate. Other patients will be asked to drink large amounts of water at regular time intervals. Adapt your model to be able to predict what will happen when the concentrations of the drugs can change due to different rates of water intake.

Problem C - Game Play

Provide an analysis of which of the possible factors of the game result in the biggest changes for small changes. For example, does a small change in the number of obstacles have a large impact on the game? Does a small change to the amount of propellant make a big difference? Use your analysis to augment your report to indicate which factors will require the most testing and