Our problem is this: A simple device will be tested in which the materials will be dropped from a great height, and a fan will blow air across the stream of falling material. Determine the minimal height and wind speed that can be used to separate 30-40 percent of the paper that is in the falling column of material.

We will assume that

- all pieces of paper and cardboard are rectangles and fall with face parallel to the ground,
- the fan spans the entire length of the drop,
- the fan speed decays exponentially and equally affects all recyclables,
- recyclables are evenly distributed between paper and cardboard,
- the trajectory of the recyclables are unaffected by other recyclables,
- recyclables are dropped from a single starting point,
- recyclables are dropped without an initial velocity, and
- recyclables range uniformly from falling perpendicular to the ground to falling parallel to the ground.

Throughout the paper we will denote the position of the paper and cardboard as a function of time in the horizontal direction as $x_p(t)$ and $x_c(t)$ respectively, and similarly denote the positions in the vertical direction as $y_p(t)$ and $y_c(t)$. Also we denote the initial fan speed as $F_s$ and gravity as $g$.

Now consider the problem where air resistance is taken into account. Recall that drag force is given by

$$F_D = \frac{1}{2} \rho v^2 AC_D$$

where $\rho$ is the density of the fluid (in this case the air), which is 1.225 kg/m$^3$, $v$ (m/s$^2$) is the speed of the object relative to the fluid or air, $A$ (m$^2$) is the cross sectional area of the object facing the direction of motion, and $C_D$ is the dimensionless drag coefficient. In the horizontal direction the speed of the object (the fan speed) is the same as the speed of the air in the horizontal direction, so there is no drag force in the horizontal direction, thus using newton’s law $F = ma$ and letting $m_1$ and $m_2$ be mass of a piece of paper and cardboard respectively, we get for each individual piece of paper and cardboard the differential equations:

$$\frac{dx_p}{dt} = F_s e^{-kt},$$
$$\frac{dx_c}{dt} = F_s e^{-kt},$$
$$m_1 \frac{d^2 y_p}{dt^2} = -m_1 g + \frac{1}{2} \rho v^2 AC_D,$$
$$m_2 \frac{d^2 y_c}{dt^2} = -m_2 g + \frac{1}{2} \rho v^2 AC_D.$$
than any other piece of cardboard. Since we assumed that \( A \) ranges uniformly from 0 to \( R \) we will assume that the pieces of cardboard land uniformly between \( c_s \) and \( c_b \). 

Now assume the density of each piece of cardboard is given by \( \rho \) in kg/m\(^2\), then for the cardboard piece \( C \), \( M = R\rho_c \). When \( A = 0 \), we have that
\[
\frac{d^2y_c}{dt^2} = -g,
\]
and when \( A = R \),
\[
\frac{d^2y_c}{dt^2} = -g + \frac{1}{2R\rho_c}\mu C_D = -g + \frac{1}{2\rho_c}\mu C_D.
\]
Notice that these differential equations are independent of mass (and hence of area)! Therefore, assuming all pieces of cardboard have a common density, each piece of cardboard will land between \( c_s \) and \( c_b \). Also the paper pieces will land between two points \( p_s \) and \( p_b \) determined similarly (notice \( p_s = c_s \) since these are both points obtained when the drag force is zero).

A sample simulation is given below for an example height and fan speed, demonstrating the range of distance that pieces of cardboard and paper can fall between.

The x axis represents varying heights of the drop, and the y axis represents various fan speeds. Each color represents the percentage of successfully sorted paper, with blue being the lowest percentage and yellow the highest.

![Range of Trajectories](image1)

Now let \( \rho_p \) be the density of paper. Notice if \( \rho_p < \rho \), then the drag force affecting paper is greater so we get that \( p_b > c_b \). Thus we can get the percentage of paper sorted by the expression \( \frac{p_b-c_b}{p_b-c_s} = \frac{p_b-c_s}{p_b-c_b} \).

From research we obtained the values \( \rho_c = 0.13 \) and \( \rho_p = 0.08 \) (kg/m\(^2\)). After running simulations on MATLAB we obtained that the minimal fan speed and height required to sort the paper 30%-40% to be 2.4m/s and 1m respectively. This can be seen in the heat map below.

![Sorting](image2)