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STUDENT VERSION
Energy in a Mass-Spring System

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STATEMENT

Background

Consider a simple mass-spring system depicted in Figure 1 where m is the mass of an object suspended by a spring. Given some initial energy or displacement in the vertical direction the mass will oscillate vertically. Using Newton's second law of motion we note immediately that the sum of the forces acting on the mass will be balanced by the product of the mass and the acceleration:

$$ma = \sum F. \tag{1}$$

There are three primary forces driving the oscillations in the mass-spring system:

- the restoring force due to the spring: F_r,
- the damping force working against the motion of the mass: F_d, and
- any external forces that may depend on time: $f(t)$.

For an ideal linear spring, Hooke's Law states that the restoring force is proportional to the displacement of the mass from equilibrium: $F_r = -ky(t)$. The proportionality constant k is called the spring constant. In simple terms, Hooke's Law states that if the mass-spring system has been stretched a long way from equilibrium then the restoring force will be large. If, however, the mass-spring system has been stretched only a short way from equilibrium then the restoring force will be small. The negative sign indicates that the force will pull in the opposite direction of the position

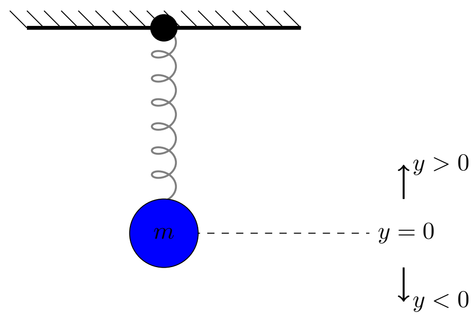

Figure 1. A mass-spring oscillating system connected to a rigid body above with mass m. The coordinate system uses $y = 0$ as the rest position of the mass with $y > 0$ indicating positions above equilibrium and $y < 0$ indicating position below equilibrium.

and, hence, back toward equilibrium. Since force is measured in Newtons, the spring constant k has units of Newtons per meter.

For an ideal linear spring, the force due to drag will oppose the motion in a manner that is approximately proportional to the velocity of the mass: $F_d = -by'(t)$. That is to say, if the mass is moving quickly then the force due to drag will be large and if the mass is moving slowly then the force due to drag will be small. The damping constant has units of Newtons per meter per second.

External forces, $f(t)$, are any other forces that act on the system. Examples of such forces would be the presence of a magnetic field, the presence of upward or downward air currents, a periodic forcing term such as pushes or pulls on the mass or spring, etc.

Using Newton's second law (1) we can write the balanced forces as

$$ma = F_r + F_d + f(t). \tag{2}$$

Substituting the restoring force, the damping force, and $a = y''$ into (2) gives the linear second-order differential equation

$$my''(t) = -ky(t) - by'(t) + f(t). \tag{3}$$

Rearranging (3) algebraically gives the standard form for a linear second-order non-homogeneous differential equation:

$$my'' + by' + ky = f(t). \tag{4}$$

It should be noted that the forms of F_r and F_d used to build (4) are idealizations. If the spring is stretched *too far*, if the speeds are *too high*, or if the materials used are atypical in some way then different forms of the restoring and damping forces may be necessary.

In this problem we investigate how the mass-spring system (4) can be described in terms of potential and kinetic energy. We begin with a few definitions:

- **Kinetic Energy**, the energy of motion, is defined as

$$E_{kinetic}(t) = \frac{\text{mass} \times \text{velocity}^2}{2} = \frac{1}{2} m \left[y'(t)\right]^2 .$$

- **Potential Energy** in a mass-spring system, also called the *elastic potential*, is defined as

$$E_{potential}(t) = \frac{\text{spring constant} \times \text{position}^2}{2} = \frac{1}{2} k \left[y(t)\right]^2 .$$

- The **Total Energy** in a mechanical system is the sum of the kinetic energy and the potential energy.

$$E_{total}(t) = E_{kinetic}(t) + E_{potential}(t)$$

The units of energy are Newton-Meters or Joules. In terms of a mass-spring system, kinetic energy is the energy that the mass has due to its motion. If the mass is at rest then the kinetic energy is zero. Also, if the mass has reached a maximum displacement (and is just about to turn around and move in the opposite direction) the kinetic energy will be zero. The potential energy in a mass-spring system is the energy that the mass has relative to its equilibrium position. If the mass is at equilibrium then it has no potential energy but if the mass is far from equilibrium it will have a large amount of potential energy.

Student Tasks:

The following tasks ask you to explore the mass-spring system by examining the total energy of the system. The tasks are necessarily open ended meaning that each group could (and should) get different answers for each task. At the end of the explorations you will write your results in a formal lab report.

1. **Make a conjecture**: In what cases (related to m, b, k, and $f(t)$) do you think that the total energy will be constant? Give a few sentences to support your claim and then create plots of position, potential energy, kinetic energy, and total energy to graphically verify your conjecture.
2. **More conjectures**: In what cases (related to m, b, k, and $f(t)$) do you think that the total energy will be decreasing, increasing, or oscillating in a mass-spring system? Give a few sentences to support your claims.
3. **Exploration:** Fully explore how the energy behaves in the mass-spring system. To make your exploration somewhat easier let's assume the following:

   $$\text{mass} = m = 1\text{kg}, \quad \text{initial position } = y(0) = 0\text{meters}, \quad \text{initial velocity} = y'(0) = 1\text{meters/sec.}$$

   This way you only have the damping constant b, the restoring constant k, and the forcing function $f(t)$ to experiment with. The given initial conditions will start the mass at equilibrium and given it an initial upward velocity.

   Use the background information presented earlier in this document to conjecture and test combinations of b, k, and $f(t)$ that result in the following situations. You must find the situations listed, and the last item in the following list gives you a chance to look for situations that are not listed.

(a) Find a combination of parameters where the total energy drops slowly to zero.

(b) Find a combination of parameters where the total energy drops very quickly to zero.

(c) Find a combination of parameters where the total energy oscillates but never reaches zero and does not increase for all time.

(d) Find a combination of parameters where the total energy increases for all time.

(e) Find a combination of parameters where the total energy changes initially but eventually finds a nonzero equilibrium.

(f) Find a combination of parameters where the system exhibits resonance (where the unforced frequency matches the frequency of the forcing term).

(g) Find a combination of parameters where the total energy oscillates with two frequencies: a slow frequency and a faster frequency (hint: get the resonant system first and then change the frequency of the forcing term).

(h) Now go find several other combinations of parameters that give behaviors different than the ones listed above.

4. **Summary:** Summarize all of your findings into a well-formated lab report clearly showing the mathematical and graphical representations all of the cases used in your experimentations. Your initial conjectures (from problems 1 and 2) may have been incorrect so take this chance to clarify what you've found. Your summary must include general descriptions of the following four general scenarios.

(a) The total energy remains constant.

(b) The total energy drops to zero.

(c) The total energy increases without bound.

(d) The total energy oscillates.