

TEACHER VERSION

Spread of a Disease

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Abstract: Many students fear the term “Differential Equations.” They think it is scary mathematics. In this project I want to use the algebra based concept “difference quotient” to solve differential equations models with the help of Excel. That means even students with only a College Algebra background, can still enjoy differential equation models. Key is the fact that the derivative of a function is the limit of difference quotient, $\frac{f(x+h)-f(x)}{h}$, of the function. If we choose h small enough, we can approximate the derivative by a difference quotient and this can be used in a differential equation.

Spread of a Disease project can be used not only to model the spread of a disease, but it can also simulate the spread of a rumor. This Sleeping Beauty project can connect classroom mathematics and real-world applications. It is a modified SIR model. It can also model disease control.

Keywords: Derivative, Difference Quotient, Proportion, Sleeping Beauty.

Tags: Spread, Contagious Disease

STATEMENT

Spread of a Disease

A contagious disease such as a flu virus is spread in a community by people encountering other people. Let $y(t)$ denote the number of people who have the disease. We call them infected. Let $x(t)$ denote the number of people who are healthy. We call them susceptibles.

It is reasonable to assume that the rate $\frac{dy}{dt}$ at which the disease spreads is proportional to the number of interactions between those two groups of people. That means $\frac{dy}{dt}$ is jointly proportional to $x(t)$ and $y(t)$. Therefore, we have the equation $\frac{dy}{dt} = kxy$, where $k > 0$ is

the proportionality constant. Likewise, $\frac{dx}{dt} = kxy$, where $k < 0$ is the proportionality constant.

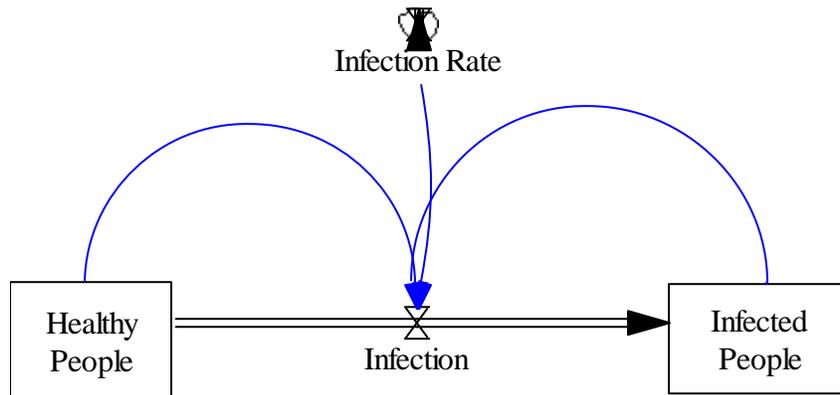


Figure 1. Showing the effect of infected people on healthy people.

The diagram in Figure 1 shows the rate of change of infected people is equal to the product of infection rate k and the number of interaction of healthy people $x(t)$ and infected people $y(t)$.

Example: Let us assume that initially $x(0) = 100$ and $y(0) = 5$, while the infection rate is $k = 0.05$.

We approximate the derivative $\frac{dy}{dt}$ by the difference quotient $\frac{y(t+h)-y(t)}{h}$, and hence

$$\frac{y(t+h)-y(t)}{h} \approx 0.05 * x(t) * y(t).$$

We solve for $y(t+h)$ and obtain the resulting formula (1) for $y(t+h)$, the number of infected at time $t+h$ in terms of both the number of susceptibles at time t , $x(t)$, and the number of infecteds at time t , $y(t)$.

$$y(t+h) \approx y(t) + h * 0.05 * x(t) * y(t). \quad (1)$$

Similarly, $\frac{dx}{dt} \approx \frac{x(t+h)-x(t)}{h} \approx -0.05 * x(t) * y(t)$ yields (2)

$$x(t+h) \approx x(t) - 0.05 * x(t) * y(t) \quad (2)$$

In this example we will let $h = 0.2$. Later, we can try several values for h to explore the differences of the solutions.

Equations (1) and (2) are the formulae that we need to enter Excel Worksheet.

Initially set up an Excel Worksheet row 1, row 2 and row 3 as follows:

- i. In row three enter initial conditions.

- ii. In cell A5, enter formula $=A4+J\$2$. In cell B4 enter formula $=\text{ROUNDDOWN}(B4-H\$2*J\$2*B4*C4,0)$. In cell C4 enter formula $=\text{ROUNDUP}(C4+H\$2*J\$2*B4*C4,0)$.
- iii. B4 and C4 are the formula (2) and (1) respectively and the Round function (ROUNDDOWN) is used to round decimals to integers.
- iv. Copy formulas of B4 and C4 down until you see 0 in the column B.
- v. Select cells A3 to C13 as data for a graph.

Figure 2 shows a sample solution using EXCEL.

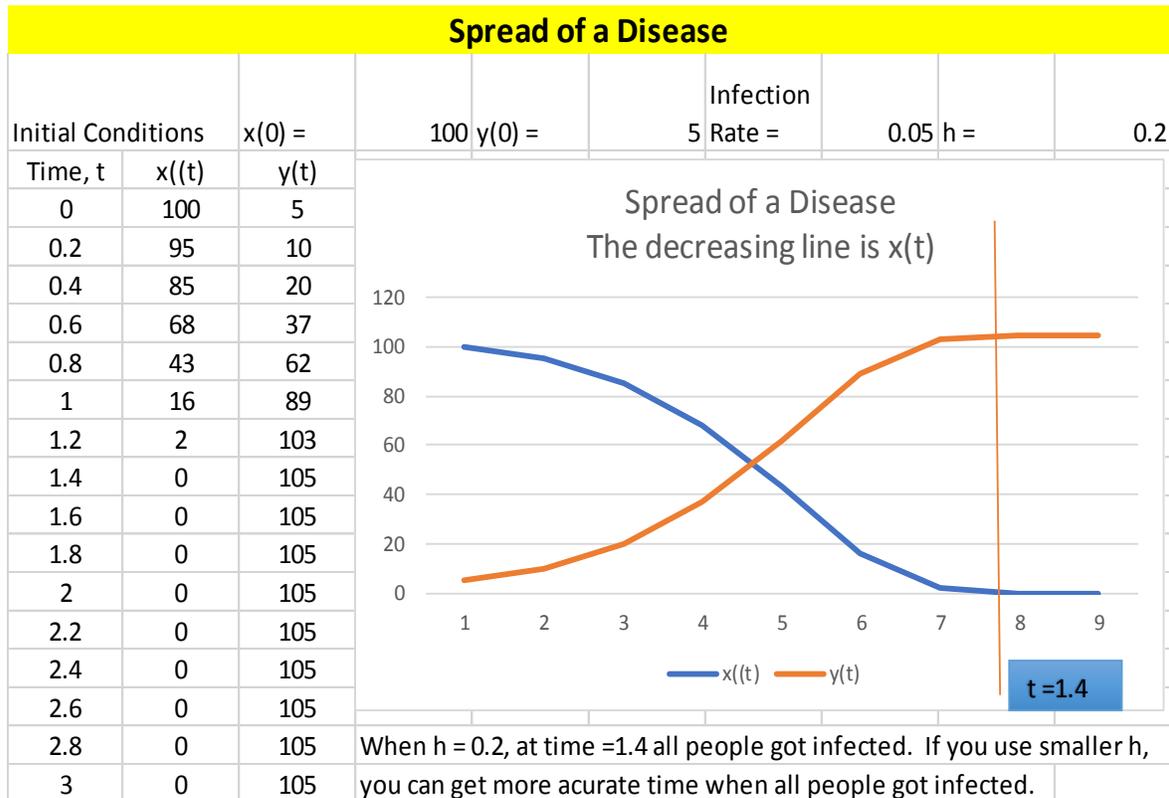


Figure 2. Sample Solution Using EXCEL.

Activities

1. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine a differential equation for the number of students, $y(t)$, who have contracted the flu if the rate of change at which the flu spreads is proportional to the number of interactions between the number of students who have the flu and the number of student who have not yet been exposed to it. Without medication when will all students have flu?

Instructions:

- (i) This is a similar question to the previous example with initial condition $x(0) = 1000$, and $y(0) = 1$.
- (ii) Submit your Excel worksheet with appropriate graph and explanations.

2. Suppose a small community has a fixed population of n people. If one person who wants to spread a rumor is introduced into this community. Let $y(t)$ denote the number of people who know the rumor and $x(t)$ denote the number of people who do not know the rumor. It is reasonable to assume that the rate $\frac{dy}{dt}$ at which the rumor spreads is proportional to the number of encounters between the two groups of people. How do we construct a differential equation to interpret this situation? Assume the proportionality constant is 0.05. How long will it take for the rumor to spread to everyone in the community?

Instructions:

- (i) This is a similar question as previous example with conditions $x + y = n + 1$, $x(0) = n$ and $y(0) = 1$. Use $n = xx00$ where xx is your last two digits of your social security number.
- (ii) Submit your Excel worksheet with graph and explanations.

3. Modified SIR Model - Sleeping Beauty [2]

We introduce a slightly different and more versatile mode, the SIR model where S stands for the number of Susceptibles; I stands for the number of infected; and R stands for the number of recovered in the population. The SIR model assumes that a disease runs its course quickly enough that births and deaths (from other causes) will not affect the population and that the disease itself will not kill off the population. The SIR model also assumes recovered persons will not infect others, or that a disease will not mutate and infect the same person multiple times.

If we assume recovered persons, R , are back to the susceptible, S , group then recovered persons have an equal chance as healthy people to get infected.

Sleeping Beauty is a well-known fairy tale. The story is that the princess (P) gets poisoned by a witch (W) and thus became Sleeping Beauty (B).

Then along comes a prince (K) who kissed sleeping beauty and turned her back to princess. In this story princess and sleeping beauty are the main variables because they had a change in quantities. The witch and prince are auxiliary variables for they are constants. The parameters are rate of poisoned by witch (r_1) and rate of kissed by prince (r_2).

The diagram illustrating these interactions is shown in Figure 2 following [2].

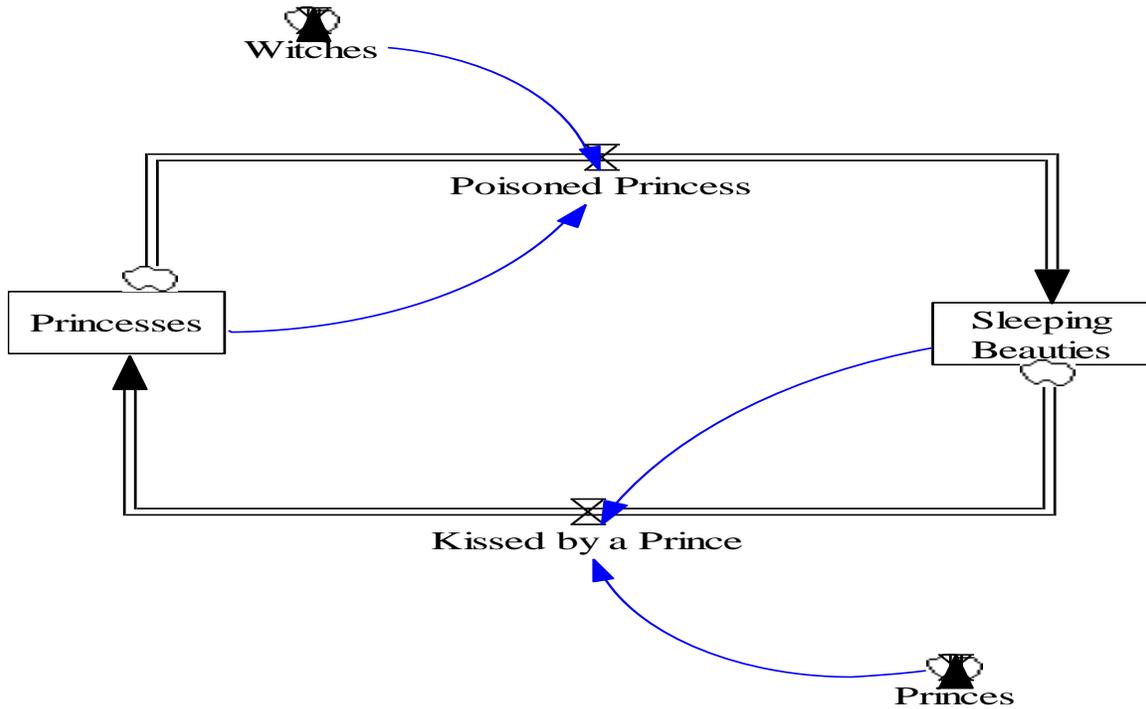


Figure 2. Showing the dynamics of the Sleeping Beauty Model.

The rate of change of Sleeping Beauty B is the difference of,

$$\frac{dB}{dt} = r_1 * W * P - r_2 * K * B$$

and the rate of change of P is

$$\frac{dP}{dt} = r_2 * K * B - r_1 * P * W.$$

Again, we approximate $\frac{dB}{dt}$ by $\frac{B(t+h)-B(t)}{h}$ and approximate $\frac{dP}{dt}$ by $\frac{P(t+h)-P(t)}{h}$. Therefore, we have the formula (3) and (4) to enter into our Excel Worksheet.

$$B(t+h) \approx B(t) + h * (r_1 * P(t) * W - r_2 * B(t) * K), \quad (3)$$

$$P(t+h) \approx P(t) + h * (r_2 * B(t) * P(t) - r_1 * P(t) * W). \quad (4)$$

Example: Here we assume initially there are 50 princesses, 2 witches, and 4 princes and further assume that the poisoned rate r_1 is 0.4 and kissed rate r_2 is 0.2. We will use $h = 1$ at this example. You may try other smaller values to explore the differences in solutions. After we implement the above formulae to Excel worksheet, we have solution as following:

- i. Initially set up an Excel Worksheet with row 1, row 2 and parameters rates as shown in Figure 3.
- ii. At row two enter initial conditions

- iii. In cell E2, enter formula =ROUND(G5*G\$2*G\$4*C2,0); At D2 enter the formula =ROUND(G5*G\$1*G\$3*B2,0). In cell C2 enter the formula =ROUND(C2+D3-E2,0). In cell B2 enter the formula =ROUND(B2+E2-D3,0). Use Excel Round function (ROUND) to round all decimal entries to integers.
- iv. C2 and B2 are the formula (3) and (4) respectively.
- v. Copy formulas of B2, C2, D2, and E2 down.
- vi. Select cells B2 to C20 as data for a graph.

Sample Excel Solution:

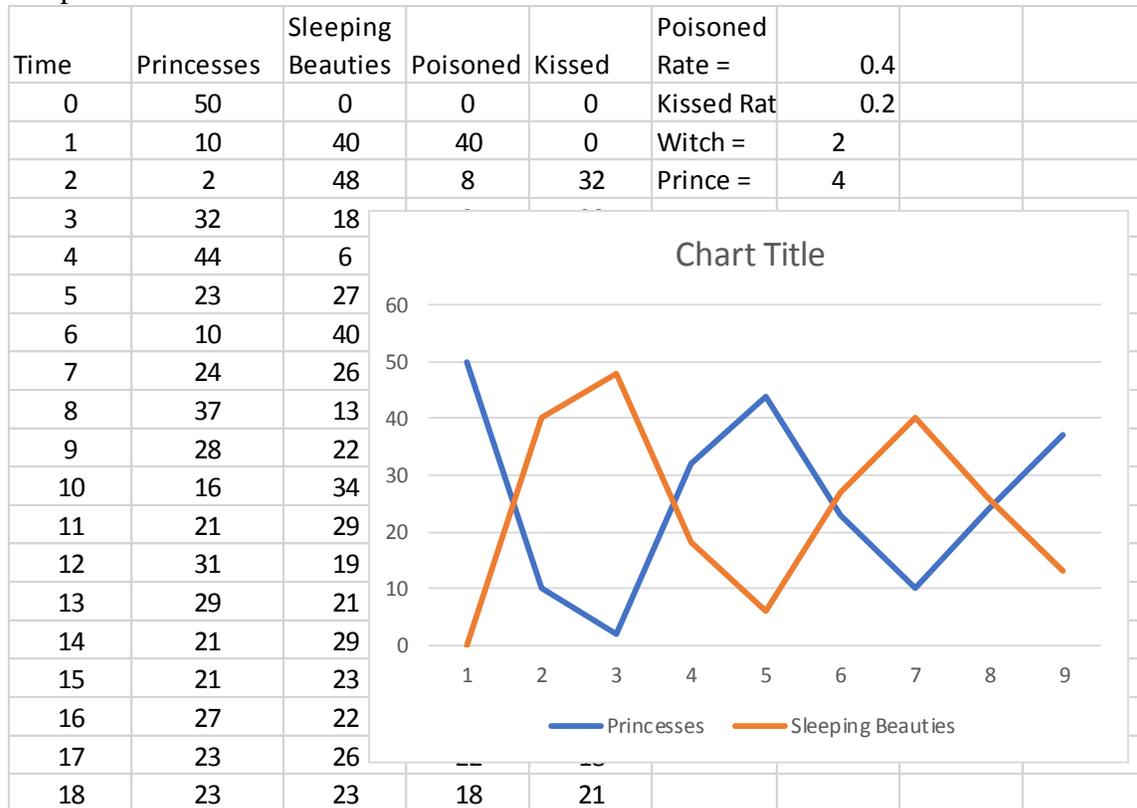


Figure 3. Excel worksheet for analysis of Sleeping Beauty Model.

Remarks:

1. You may want to use smaller h values to explore the differences in the solutions.
2. Because of rounding error, you may have negative numbers for princesses or negative numbers for sleeping beauties. How would you explain those negative numbers? What could you do to eliminate them?
3. If we replace princesses by healthy people in a community, witches by virus, sleeping beauties by infected patients, and princes by medical treatments, it is a simple model of disease control. Certainly, you can add more conditions to make the model be a better representation of real situations.

Activity:

4. Suppose three students carrying a flu virus return to an isolated college campus of 1,000 student. The infection rate is 0.4. There is a 60% chance students can get well if students get doctor's treatment. How many doctors do we need to control the outbreak of the flu?

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Excel Tutorials:

At http://www.gravenhurst.ca/en/library/resources/Excel_Handout.pdf one can find a Microsoft Excel Tutorial Handout which students will find useful [1].

At <https://support.office.com/en-us/article/add-a-scroll-bar-or-spin-button-to-a-worksheet-f8443be3-ff00-4cad-bb2f-bf0f88ebf5bb> on can find a tutorial on how to add a Scroll bar or spin button to a worksheet [2].

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