www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION

Solow-Swan Model of Economic Growth

Yuri Yatsenko
Management, Marketing, and Business
Houston Baptist University
Houston TX USA

STATEMENT

Mathematical modeling with differential equations is widely used in all sciences and economics is not an exception. The goal of this project is to learn how to construct and investigate a simple dynamic model, which plays an important role in the economic growth theory (it is known as *the Solow-Swan model of economic dynamics)*.

The primary submission is a hard copy report that should include this completed assignment (you can print it and fill in empty spaces) and offer a short summary of what you have learned. This can be an individual or group project.

PART 1. Construct a mathematical model of linked production and economic processes described below.

We consider an economy (usually, a separate country) that uses the labor resource $L(t)$ and production capital $K(t)$ to produce a certain output $Q(t)$ of a uniform aggregated product (say, GDP) at any time t. A constant portion of the output $Q(t)$ is used for consumption $C(t)$ and the rest for the investment $I(t)$ into new production capital.

The dynamics of this economic system in the continuous time t can be described by FIVE dynamic characteristics (unknown variables):

- $Q(t)$ - the total output produced at time t,
- $C(t)$ - the amount of consumption,
- $I(t)$ - the amount of gross investment into capital,
- $L(t)$ - the amount of labor,
- $K(t)$ - the amount of capital.

To build the model, we make five natural economic assumptions, which are also illustrated in the flow diagram in Figure 1.

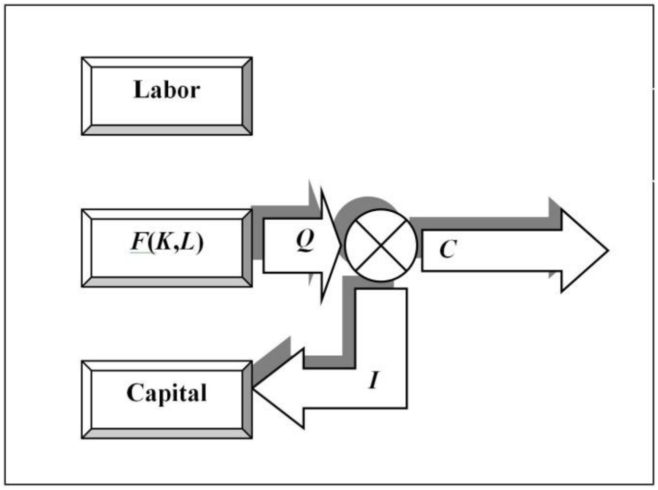

Figure 1**.** The flow diagram of the Solow-Swan model

You are not asked to justify those assumptions, just accept them and use them in the construction of the model.

Assumption 1. The product output $Q(t)$ is determined by the production function $F(K(t),L(t))$ that describes the production process.

Assumption 2. After production, the output $Q(t)$ is split between the consumption $C(t)$ and the investment $I(t)$ into new production capital.

Assumption 3. The investment $I(t)$ takes a constant portion s, $0 < s < 1$, of the output $Q(t)$ of the total product. The portion s is usually referred as the *saving rate* and assumed to be constant in our model.

Assumption 4. The rate of change of the capital $K(t)$ increases with the investment $I(t)$ but decreases with the depreciation of the capital at a constant rate $\mu > 0$. Depreciation means that a constant fraction of the capital leaves production process at each point of time.

Assumption 5. Finally, we assume that the labor $L(t)$ grows exponentially at a constant exogenous rate η starting with a given initial value L_0.

Correspondingly, the constructed model should contain FIVE equations to describe:

1) the production of the output $Q(t)$,
2) the distribution of the output $Q(t)$,
3) the dynamics of the investment $I(t)$,

4) the dynamics of change of the capital $K(t)$, and
5) the dynamics of the labor $L(t)$.

The last two equations shall be presented by means of differential equations.

Solution. In accordance with Assumptions 1-5, the model of the economic system under study is described by the following equations:

$$Q(t) = \underline{\qquad\qquad\qquad} , \tag{1}$$

i.e. the output Q is determined by the production function $F(K,L)$;

$$Q(t) = \underline{\qquad\qquad\qquad} , \tag{2}$$

i.e. the output Q is distributed between the consumption C and the investment I;

$$I(t) = \underline{\qquad\qquad\qquad}, \tag{3}$$

i.e. the saving rate $s(t) = I(t)/Q(t)$ is assumed to be constant;

$$K'(t) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}, \tag{4}$$

i.e. the capital K depreciates at a constant rate $\mu > 0$ (a constant fraction of the capital leaves production at each point of time); and

$$L'(t) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}, \tag{5}$$

i.e. the labor $L(t)$ grows at a constant exogenous rate η.

The constructed model (1)-(5) is one of the most celebrated models in economic growth theory and is known as the *Solow-Swan Model*. It is named after the economists R. Solow and T. Swan. Robert Solow was awarded the Nobel Prize in Economics in 1987 for his contribution to economic sciences, which includes this model and its generalizations. The model (1)-(5) of economic dynamics has become a foundation for many further successful studies. In particular, more advanced models consider the variable saving rate $s(t)$ as an endogenous control function.

PART 2. Provide a qualitative steady-state analysis of the Solow-Swan model (1)-(5).

The steady-state analysis plays an important role in economics and is mathematically simpler than a complete dynamic analysis. The goal of a steady-state analysis is to find possible *steady states*, which can be

- a *stationary trajectory* (a constant solution) or,
- a *balanced growth trajectory* (a balanced growth path).

A stationary trajectory assumes all unknown variables to be constant in time, while all variables grow at the same constant rate along a balanced growth path.

The presence of steady states often indicates the quality and suitability of mathematical models employed in economics.

Our objective in this part is to find possible steady states in a constructed model under the following assumption.

Assumption 6: The production function $F(K,L)$ is linearly homogeneous, i.e. it can be written as $F(K,L) = LF(K/L,1) = Lf(k)$ where $k = K/L$ and the function $f(k) = F(k,1)$ has the following properties:

$$f(0) = 0, \qquad f'(k) > 0, \qquad \lim_{k\to 0} f'(k) = \infty, \quad \text{and} \quad \lim_{k\to\infty} f'(k) = 0, .$$

The new unknown variable $k = K/L$ is known as the *capital-labor ratio*. Such production functions are known as neoclassical production functions in economic theory.

In order to find a balanced growth path, we will assume that all unknown functions in the model (1)-(5) increase with the same rate η as labor, $L(t)$, increases, that is,

$$Q(t) = \bar{Q}\; e^{\eta t}, \quad K(t) = \bar{K}\; e^{\eta t}, \quad I(t) = \bar{I}\; e^{\eta t}, \quad Q(t) = \bar{Q}\; e^{\eta t}, \quad C(t) = \bar{C}\; e^{\eta t}, \tag{6}$$

and find the unknown constants $\bar{K}$, $\bar{I}$, $\bar{Q}$, and $\bar{C}$. The constants $\bar{K}$, $\bar{I}$, $\bar{Q}$, and $\bar{C}$ are often called in economics the *level variables*.

Formally, a stationary trajectory is a special case of the balanced growth (6) at $\eta = 0$. To find a stationary trajectory, we would simply assume all variables to be independent of time t.

Step 1. Model Reduction.

Here we will show that the nonlinear system of five equations (1)-(5) under Assumption 6 can be reduced to one nonlinear differential equation.

Because the production function $F(K,L)$ is linearly homogeneous, then $F(K,L) = Lf(k)$. Substituting this formula into the (1) and, next, substituting $Q(t)$ from (1) into (3) and $I(t)$ from (3) into (4), we obtain

$$K'(t)/L(t) = \underline{\qquad\qquad\qquad}, \tag{7}$$

where

$$k(t) = K(t)/L(t) \tag{8}$$

is the unknown capital-labor ratio. On the other side, taking the derivative of (8), we have

$$k' = \frac{K'L - KL'}{L^2} = \frac{K'}{L} - \frac{K}{L} \times \frac{L'}{L}. \tag{9}$$

Using (5) in (9), we obtain

$$k'(t) = \underline{\hspace{6cm}}. \tag{10}$$

Finally, combining the equalities (7) and (10) and excluding the term $K'(t)/L(t)$, we obtain the following equation with respect to one unknown variable $k(t)$ only:

$$k'(t) = \underline{\hspace{6cm}}. \tag{11}$$

It is called the ***fundamental equation of the Solow-Swan model***. Thus, the dynamics of the differential model (1)-(5) is reduced to one *nonlinear* differential equation (11) with respect to the unknown capital-labor ratio $k(t)$. The differential equation (11) is *autonomous* (its coefficients do not explicitly depend on t), which simplifies its analysis.

Step 2. Steady-State Analysis

Let us analyze the possibility of a balanced growth trajectory in the model (1)-(5). It is easy to see that the original variables, $Q(t)$, $C(t)$, $I(t)$, and $K(t)$, of the model grow with the same rate if the capital–labor ratio, $k(t)$, is constant. Indeed, substituting k as a constant into (3), (4), and (5), we obtain

$$K(t) = kL(t), \;\; I(t) = (\mu + \eta)K(t), \;\; Q(t) = (\mu + \eta)K(t)/s, \;\; C(t) = Q(t)-I(t), \tag{12}$$

i.e., all these functions increase with the same rate η as the labor $L(t)=L_0\exp(\eta t)$. Therefore, to find steady states, we should assume $k(t)$ is constant. Then $k'(t) = 0$ and the nonlinear differential equation (11) is reduced to the nonlinear equation

$$sf(k) = \underline{\hspace{4cm}} \tag{13}$$

for possible steady states k is constant. By Assumption 6, $f(0) = 0$, $f'(k) > 0$, $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k) = 0$, therefore, (13) has a unique solution $\hat{k} = \hat{k}(s)$, a non-negative constant, for any given value $s > 0$. The solution $\hat{k}$ is the intersection point of the increasing concave curve $f(k)s$ and the straight line $(\mu+\eta)k$ for any fixed $s > 0$. The steady-state capital-labor ratio $\hat{k}(s)$ increases when the saving rate s increases.

After we found $\hat{k}(s)$, the corresponding balanced growth path, $Q(t)$, $C(t)$, $I(t)$ and $K(t)$, of the model is calculated using (12) at $k = \hat{k}(s)$ and grows with the same rate η.

PART 3. Perform Static Optimization in the Solow-Swan model.

A natural management goal for any economic system (a country, region, or a firm) is to find the rational (or optimal) management strategy that maximizes its profit, or minimizes expenses, or renders similar optimization goals.

In the Solow-Swan model (1)-(6), we can find the optimal saving rate s^* to be constant, and the corresponding steady-state, $k^* = \hat{k}(s^*)$, such that maximize the steady-state consumption per capita. Moreover, this optimization problem can be solved using elementary techniques of calculus.

Static Optimization Analysis.

For a given saving rate s, the steady-state consumption per capita $c = C/L$ is determined by the formula,

$$c(s) = f(\hat{k}(s)) - (\mu + \eta)\hat{k}(s), \tag{14}$$

where the corresponding steady-state capital-labor ratio $\hat{k}(s)$ is determined by the nonlinear equation (13). Because $f(0) = 0$, $f'(k) > 0$, and $f''(k) < 0$, the composite function (14) increases for smaller values of s and decreases for larger s.

Then, we can determine the optimal saving rate s^* to be constant and the corresponding steady-state $k^*=\hat{k}(s^*)$ that maximize the consumption per capita (14) from the following optimization problem:

$$\max_{0<s\le 1} c(s) = \underline{\hspace{4cm}}. \tag{15}$$

The maximization of (15) is an optimization problem with one scalar variable s. Following the First Derivative Test, the necessary extremum condition for (15) at an interior $0 < s < 1$ is $c'(s) = 0$. Taking the derivative of the composite function in (14), we have

$$d\,[f(\hat{k}(s)) - (\mu + \eta)\hat{k}(s)]/ds = \underline{\hspace{4cm}} = 0. \tag{16}$$

As a solution of the equation (16), the optimal capital-labor ratio k^* should satisfy

$$f\,'(k^*) = \underline{\hspace{2cm}}\ . \tag{17}$$

The formula (17) is known as the ***golden rule of capital accumulation***. It implies that the rate $f\,'(k)$ of production increase at the optimal value k^* should be equal to the sum $\mu + \eta$ of the depreciation and the labor growth rates. The rate of product increase with respect to capital is known in economics as the *marginal product of capital*.

After determining the optimal k^* from (17), the corresponding ***golden-rule saving rate*** is found from (13) as

$$s^* = \underline{\hspace{3.5cm}}. \tag{18}$$

The formulas (13) and (18) for the optimal s^* and k^* are known as the ***golden rule of economic growth.***

Example: A frequently used special form of production function is the *Cobb-Douglas production function* $F(K,L) = AK^{\alpha}L^{1-\alpha}$, $0 < \alpha < 1$. Then, the function $f(k)$ is $f(k) = Ak^{\alpha}$ and the *golden rule of economic growth* is

$$s^* = \_\_\_\_\_, \qquad k^* = \_\_\_\_\_\_\_\_\_\_\_\_. \tag{19}$$

At the optimal steady state (s^*, k^*) and the given labor $L(t) = \bar{L}e^{\eta t}$, the balanced growth trajectory $Q(t)$, $C(t)$, $I(t)$ and $K(t)$ of the model (1)-(6) is

$$K(t) = \bar{K}\, e^{\eta t}, \quad I(t) = \bar{I}\, e^{\eta t}, \quad Q(t) = \bar{Q}\, e^{\eta t}, \quad C(t) = \bar{C}\, e^{\eta t}, \tag{20}$$

where

$$\bar{K} = \bar{L}\,k^*, \quad \bar{I} = (\mu + \eta)\bar{L}\,k^*, \quad \bar{Q} = \frac{(\mu+\eta)\bar{L}k^*}{s}, \quad \bar{C} = \frac{(1-s)(\mu+\eta)\bar{L}k^*}{s}. \tag{21}$$

Because the balanced output Q, consumption C, investment I, and capital K increase with the same rate η as the exogenous labor L, the Solow-Swan model is classified in the economic theory as the *exogenous growth model*.

In the special case of constant labor $L(t) = \bar{L}$ (i.e. at $\eta = 0$), the steady state (20)-(21) is a *stationary point*.

Additional historical information. Robert Merton Solow (born in 1924) is an American economist, who for his contribution to the theory of economic growth was awarded the John Bates Clark Medal in 1961, the Nobel Memorial Prize in Economic Sciences in 1987, and the Presidential Medal of Freedom in 2014. Three of his PhD students also received Nobel Memorial Prizes in Economic Sciences. Trevor Winchester Swan (1918 – 1989) was an Australian economist known for his work in neoclassical growth theory and the Swan Diagram. Robert Solow and Trevor Swan published their pioneering results on the growth theory (the Solow–Swan growth model) independently and almost simultaneously.

REFERENCES

Hritonenko, N., and Y. Yatsenko. 2013. *Mathematical Modeling in Economics, Ecology and the Environment. Second Edition*, Springer: New York.