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The Poisson Process and Waiting Time

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STATEMENT

Poisson Process

Suppose you want to find the probability of 5 costumers entering a certain bank between 9 AM and 11 AM. The process that addresses this problem, and similar problems, is called the *Poisson process*. In this activity you will find a formula for the Poisson process by solving a system of differential equations. Let

$$\begin{cases} N(t) = \text{ number of people entering a bank in the time period } [0,t] \\ p_n(t) = \Pr(N(t) = n) = \text{ probability of } n \text{ people entering the bank in the time period } [0,t] \end{cases}$$

Suppose we know that λ people (on average) enter the bank in the period $[0, 1]$ (unit period of time). We then can make the following reasonable assumptions:

$$\Pr(N(\tau) = 1) = \lambda \cdot \tau \quad \text{and} \quad \Pr(N(\tau) \geq 2) = 0 \quad \text{for a small } \tau$$

and the number of people arriving in any two disjoint periods of time are independent. The first two imply assumptions that

$$\Pr(N(\tau) = 0) = 1 - \lambda \cdot \tau\,.$$

Now we have

$$\begin{aligned}\Pr(N(t+\tau)=n+1) &= \sum_{k=0}^{n+1}\left(\Pr(N(t)=n+1-k)\cdot\Pr(N(\tau)=k)\right)\,,\\ &= \Pr(N(t)=n+1)\cdot\Pr(N(\tau)=0)+\Pr(N(t)=n)\cdot\Pr(N(\tau)=1)\,,\\ &= \Pr(N(t)=n+1)\cdot(1-\lambda\tau)+\Pr(N(t)=n)\cdot(\lambda\tau)\end{aligned}$$

which implies that

$$p_{n+1}(t+\tau) = (1-\lambda\cdot\tau)\cdot p_{n+1}(t)+\lambda\cdot\tau\cdot p_n(t)\,.$$

From this it follows that,

$$\frac{p_{n+1}(t+\tau)-p_{n+1}(t)}{\tau} = -\lambda\cdot(p_{n+1}(t)-p_n(t))\,.$$

Now taking the limit, i.e. letting $\tau\to 0$ we obtain,

$$\begin{aligned}\frac{dp_{n+1}(t)}{dt} &= -\lambda\cdot(p_{n+1}(t)-p_n(t)) \text{ for } n=0,1,\cdots\\ \frac{dp_0(t)}{dt} &= -\lambda\cdot p_0(t)\,.\end{aligned}$$

Problem 1: Solve the above system of first order differential equations. The unknowns are $p_k(t)$ for $k=0,1,\cdots,n+1$.

Problem 2: Given that the average number of costumers entering a bank during 8 hrs on a weekday is 128 find the probability that the bank will have 10 costumers ernter the bank between 9 AM and 10 AM (time period $[0,1]$) on a weekday?

Waiting Time

Suppose you are observing the costumers in a bank and one costumer arrives. You want to find the probability that it takes more than certain amount of time, t, for the next customer to arrive. That is, you want to find $\Pr(T>t)$ where T is called the *waiting time*. The longer you wait the more you expect to see a costumer. Therefore $\Pr(T<t+\tau)>\Pr(T<t)$ which implies $\Pr(T>t+\tau)<\Pr(T>t)$ and we have,

$$\begin{aligned}\Pr(T>t)-\Pr(T>t+\tau) &= \Pr(t<T\le t+\tau)\\ &= \Pr(\text{one person arrives in the period } [0,\tau] \text{ and } T>t)\\ &= \Pr(N(\tau)=1)\cdot\Pr(T>t)=\lambda\cdot\tau\cdot\Pr(T>t)\,.\end{aligned}$$

Therefore

$$\frac{\Pr(T > t + \tau) - \Pr(T > t)}{\tau} = -\lambda \cdot \Pr(T > t)$$

and if we let $\tau \to 0$ then it follow that,

$$\frac{d\Pr(T > t)}{dt} = -\lambda \cdot \Pr(T > t)\,.$$

Now consider the following problems:

Problem 3: Solve the differential equation for $G(t) = \Pr(T > t)$ with the initial condition $f(0) = \Pr(T > 0) = 1$.

Problem 4: Find the cdf of the distribution of T, i.e. $F(t) = 1 - G(t)$.

Problem 5: Find the pdf of the distribution of T by differentiating its cdf, i.e. $f(t) = F'(t)$.

Problem 6: Find the average value of T or average waiting time until a customer arrives, i.e. $\displaystyle\int_0^\infty tf(t)\ dt$.

Problem 7: Suppose, on average, a bank has 5 costumers per hour. Use the previous problem to find the average waiting time. Does the answer make sense intuitively?

Problem 8: Following problem 5, suppose at about 9:30 AM you see one costumer, what is the probability that the next costumer arrives in the next 10 minutes?

Problem 9: If you see the first costumer at 9:05 AM what is the probability that you see the next costumer after 5 minutes?