



STUDENT VERSION

The Next Time You Play HvZ, Think About Differential Equations

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STATEMENT

Invented in 2005 at Goucher College (Towson, Maryland USA), Humans vs. Zombies, or **HvZ**, is a game of tag predominantly played at US college campuses. All players begin as humans, and one is randomly chosen to be the “Original Zombie”. The Original Zombie tags human players and turns them into Zombies. Zombies must tag and eat a human every 48 hours or they starve to death and are out of the game. The Zombies win when all human players have been tagged and turned into Zombies; the Humans win by surviving long enough for all of the Zombies to starve. The game rules are outlined on the game official site [2]. Many alternate rule sets can be found on the internet.

Can we use mathematical models and differential equations to study the game?

Let us first examine differential equation models in epidemiology. Epidemiology is the study of the distribution and causes of disease in populations. The spread of infectious diseases, such as measles and malaria, can be modeled by a system of non-linear differential equations. The simplest model is the **SIR** model (Susceptible-Infected-Recovered) [1]. The model makes several basic assumptions:

- A1. The total population is constant.
- A2. Once an individual has been infected and subsequently recovered, that individual cannot be re-infected, such as in the case of measles, mumps, and smallpox.
- A3. The rate of transmission of the disease is proportional to the number of encounters between susceptible and infected individuals.

A4. The rate at which infected individuals recover is proportional to the number of infected.

Define the variables:

$S(t)$ = the number of susceptible individuals at time t ,

$I(t)$ = the number of infected individuals at time t ,

$R(t)$ = the number of recovered individuals at time t .

The SIR model is illustrated schematically in Figure 1, and described by the non-linear system below, where α is the recovery rate and β is the transmission rate.

$$S'(t) = -\beta SI \quad (1)$$

$$I'(t) = \beta SI - \alpha I \quad (2)$$

$$R'(t) = \alpha I \quad (3)$$

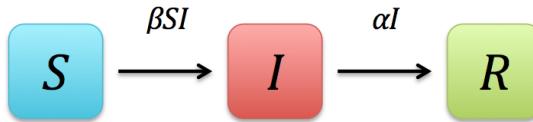


Figure 1. SIR model.

In Mathematica, you may plot the solutions and visualize the interaction of the three groups. The solutions are shown in Figure 2 for a time period of 80 days and a total population of 1000. The codes are provided in [3].

The following activities will guide you in modeling HvZ.

1. In the SIR model, why is it sufficient to consider the two-variable system consisting of (1) and (2)?
2. Find the S -nullclines (the set of points where $S'(t) = 0$) and I -nullclines (the set of points where $I'(t) = 0$) in the phase plane. Sketch them by hand in the SI -plane. Solve for the equilibria and label them in the SI -plane. Sketch the direction field, i.e., arrows indicating the direction of the solution curve. What behavior do you observe?
3. Recovered individuals may lose their immunity and become re-infected, such as in the case of malaria and tuberculosis. We add a new assumption.

A5. Re-infection occurs at a rate proportional to the population of recovered individuals.

- (a) Sketch a schematic diagram similar to Figure 1. Set up this modified model – the **SIRS** model.

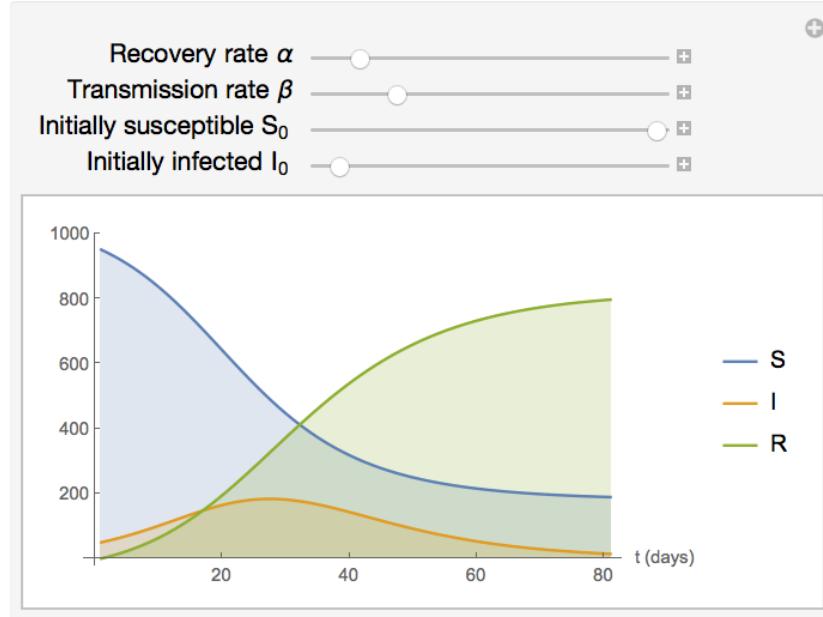


Figure 2. SIR model solutions ($S + I + R = 1000$).

- (b) Visualize the solutions in Mathematica or other softwares similar to Figure 2. Explain the difference in the behaviors of the solutions.
4. Now let's play **HvZ!**
- (a) What reasonable assumptions would you make to help set up the model for the **HvZ** game?
 - (b) Define our variables:

$S(t)$ = the number of susceptible, or humans, at time t ,

$Z(t)$ = the number of Zombies at time t ,

$R(t)$ = the number of removed individuals at time t .

One SZR model is described below, adapted from [4]. Note that the total population does not change. A schematic diagram for the interaction is shown in Figure 3.

$$S'(t) = -\beta SZ - \mu S \quad (4)$$

$$Z'(t) = \beta SZ + \gamma R - \alpha SZ \quad (5)$$

$$R'(t) = \mu S + \alpha SZ - \gamma R \quad (6)$$

Set up your SZR models based on your assumptions and sketch a corresponding diagram similar to Figure 3.

5. Suppose that at your school, 300 students signed up to play **HvZ** for a period of 7 days. They started out with 299 humans and 1 zombie. The human or zombie group with a larger population wins the game.

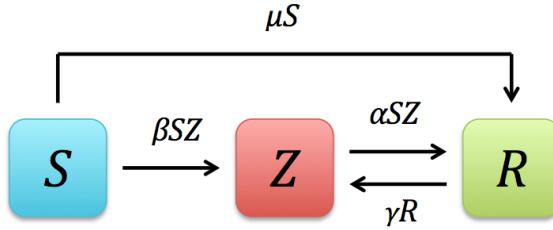


Figure 3. One SZR model.

- (a) Let's first use the system as described in (4)-(6). Write down the initial conditions for this model.
- (b) Set parameters $\alpha = 0.0015$, $\beta = 0.005$, $\gamma = 0.001$, $\mu = 0.002$.
Solve for the numerical solutions and plot them in Mathematica or other softwares.
Who won? Humans or Zombies?
- (c) What happens if you increase or decrease β ? For example, $\beta = 0.006$? 0.004 ?
- (d) You may also change the other parameter values to investigate their effects on the outcome.

If using Mathematica consider these commands “`NDSolve`” and “`Plot`”:

<http://reference.wolfram.com/language/ref/NDSolve.html>
<http://reference.wolfram.com/language/ref/Plot.html>

- 6. Now use your own SZR models. Set your parameter values and determine the game outcomes.

You may further explore modeling other pandemics such as influenza and Ebola, and other models such as MSIR, SEIR, SEIS, SEAIR, MSEIRS, etc.

The next time you play **HvZ**, think about differential equations!

REFERENCES

- [1] Hirsch, M. W., S. Smale, and R. L. Devaney. 2012. *Differential equations, dynamical systems, and an introduction to chaos (3rd edition)*. New York: Academic Press.
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- [3] Kaurov, V. 2014. Modeling a pandemic like Ebola with the Wolfram Language. *Wolfram*. <http://blog.wolfram.com/2014/11/04/modeling-a-pandemic-like-ebola-with-the-wolfram-language>. Accessed 3 September 2018.
- [4] Munz, P., O. Hudea, J. Imad, and R. J. Smith. 2009. When zombies attack!: mathematical modelling of an outbreak of zombie infection. *Infectious disease modelling research progress*.

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