

STUDENT VERSION

Separation of Variables

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INTRODUCTION

We discuss strategies to solve first order, ordinary differential equations with mathematical models when the variables may be separated. Many modeling situations which lead to such equations:

M1. Exponential growth or decay

$$\frac{dy}{dt} = ky(t)$$

M2. Logistic

$$\frac{dy}{dt} = ry(t) \frac{K - y(t)}{K}$$

M3. Extinction population levels

$$\frac{dy}{dt} = ry(t) \frac{K - y(t)}{K} \frac{y(t) - M}{M}$$

M4. Logistic with harvesting

$$\frac{dy}{dt} = ry(t) \frac{K - y(t)}{K} - h$$

M5. Gompertz

$$\frac{dy}{dt} = -ry(t) \log \left(\frac{y(t)}{M} \right)$$

where if we substitute $x(t) = \log \left(\frac{y(t)}{M} \right)$ we obtain the differential equation $x'(t) = -kx(t)$

M6. Spread of technology

$$\frac{ds}{dt} = rs(t)(K - s(t))$$

M7. Newton's Law of Cooling

$$\frac{dT}{dt} = k(T_{\text{Env}} - T(t))$$

M8. Tumor growth

$$\frac{dV}{dt} = \lambda V(t)$$

M9. Lambert's Absorption Law for light beam intensity

$$\frac{dI}{dt} = \mu I(t)$$

M10. Falling bodies with resistance

$$m \frac{dv}{dt} = -cv(t) - mg$$

M11. Mixing problems

$$\frac{dA}{dt} = A_{\text{Rate In}} - \alpha A(t)$$

M12. Chemical kinetics

$$\frac{dA}{dt} = -kA(t)^r$$

M13. Time for ant to build a tunnel

$$\frac{dT}{dx} = \alpha x$$

M14. Torricelli's Law

$$\frac{dh}{dt} = -\frac{k}{A} \sqrt{h(t)}$$

M15. Sublimation

$$\frac{dm}{dt} = -km(t)^r$$

M16. Electric circuit

$$L \frac{dI}{dt} + RI(t) = E_{\text{Emf}}$$

M17. Constant infusion tracer analysis

$$\frac{dq_1}{dt} = R_{\text{in}} A_0 - k_1 q_1(t)$$

M18. Sterile male mosquito release

$$\frac{dN}{dt} = N(t) \left(b \frac{N(t)}{N(t) + S} - d \right)$$

M19. Spread of oil slick

$$\frac{dS}{dt} = -\alpha S(t) + \beta.$$

We introduce a straightforward strategy for solving. In practice we recommend the use of computers to solve these differential equations, e.g., `DSolve` from Mathematica. Nevertheless, we believe the solution strategy we present, known as *separation of variables* gives an idea of how to build solutions and will help in understanding the nature and form of the solution, be it from pencil and paper or computer.

SEPARATION OF VARIABLES

Suppose a differential equation has the form:

$$\frac{dy}{dt} = y'(t) = F(t, y(t)). \quad (1)$$

Further, suppose (1) can be written as in (2), where the right hand side is the product of two functions, say $f(t)$ and $g(y(t))$, one a function of the independent variable, t , and the other a function of the dependent variable, $y(t)$. We can separate the variables, i.e., place all of the functions of $y(t)$ on the one side with the derivative term, $y'(t)$, and all the functions of t on the other side. Using the Chain Rule, we can reverse the differentiation process on the side with $y'(t)$ and also on the other side. Recall that these two sides of our equation will then differ by a constant. We illustrate this in (2) and we produce (3).

$$\frac{dy}{dt} = F(t, y(t)) = f(t) * g(y(t)) \quad (2)$$

Now after division of both sides by $g(y(t))$ this leads to

$$\frac{1}{g(y(t))} \frac{dy}{dt} = f(t). \quad (3)$$

While there is no certainty (consider the many possibilities for the function g) it just might be possible to antidifferentiate the left hand side using the Chain Rule and the right hand side as well.

$$\int \frac{1}{g(y(t))} \frac{dy}{dt} dt = \int f(t) dt + c \quad (4)$$

Let us consider an example in (5) from chemical kinetics, where $g(y(t)) = y(t)^2$ and $f(t) = -k$,

$$\frac{dy}{dt} = -ky(t)^2. \quad (5)$$

If we divide both sides of (5) by $g(y(t)) = y(t)^2$ then we have (6),

$$\frac{1}{y(t)^2} \frac{dy}{dt} = f(t) = -k. \quad (6)$$

Now (6) presents two functions of t , one on the left in terms of $y(t)$, our unknown function for which we are trying to solve, and $f(t) = -k$, a function of t , only – with no $y(t)$ terms. Recall, that if two functions of t are equal then their antiderivatives or indefinite integrals with respect to t differ by a constant. So integrating both sides brings us (7).

$$\int \frac{1}{y(t)^2} \frac{dy}{dt} dt = \int -k dt + c \quad (7)$$

Now what we have to do is integrate the left side and the right side of (7), both with respect to t , and then extricate $y(t)$ from whatever our result is on the left side to solve, finally, for $y(t)$. As to the constant of integration c we can determine its value by using initial value information, $y(0) = y_0$, at any point in our analysis.

There are different notations presented for this process and we present one here:

$$\int_{t=0}^{t=T} \frac{1}{g(y(t))} \frac{dy}{dt} dt = \int_{t=0}^{t=T} f(t) dt \quad (8)$$

wherein we have a definite integration in which we eliminate the need for an intermediate constant of integration c by going directly to a definite integral over the intervals $[0, T]$. This will permit us to take our next step from which we can extricate $y(T)$ or actually $y(t)$, for the form of our function y (whatever) is of final importance and so we should not be confused by t or T in the final analysis.

Another notation often presented is one using differentials, i.e., in (4) we replace $\frac{dy}{dt} dt$ with dy and we formally integrate the left hand side of (4), namely

$$\int \frac{1}{g(y(t))} \frac{dy}{dt} dt = \int \frac{1}{g(y)} dy,$$

with respect to y . Upon completion of this integration task we replace y by $y(t)$ and proceed as before. We perform the integration of the right side of (4) with respect to t and add a constant of integration and ready ourselves to solve the resulting equation for $y(t)$.

Let us see this in practice with comments at each step.

Chemical Kinetics Modeling

A model used in the study of chemical decomposition reactions is (M12). This is called a *second order reaction*. Let us suppose $k = .013$ while $r = 2$. Thus we have

$$\frac{dA}{dt} = -0.013A(t)^2, \quad A(0) = A_0. \quad (9)$$

Upon separating our variables we produce

$$A(t)^{-2} \frac{dA}{dt} = \frac{1}{A(t)^2} \frac{dA}{dt} = -0.013. \quad (10)$$

Now we can formally integrate both sides of (10) with respect to t to obtain

$$-\frac{1}{A(t)} = -A(t)^{-1} = \int A(t)^{-2} \frac{dA}{dt} dt = \int \frac{1}{A(t)^2} \frac{dA}{dt} dt = \int -0.013 dt + c = -.013t + c. \quad (11)$$

From (11) we can obtain c :

$$-\frac{1}{A(0)} = -.013(0) + c = c. \quad (12)$$

Substituting for c back in (11) yields

$$-\frac{1}{A(t)} = -.013t - \frac{1}{A(0)}. \quad (13)$$

With some algebra in (13), we have, finally,

$$A(t) = \frac{1}{.013t + \frac{1}{A(0)}} = \frac{A(0)}{0.013tA(0) + 1} \quad (14)$$

In Figure 1 we see a plot of our solution (14) to the differential equation model (9) with $A(0) = 100$ and it appears reasonable, in that the chemical is disappearing at a high rate when there is a lot of the chemical, due to our rate of change in $A(t)$ being proportional to the square of the amount of $A(t)$, $\frac{dA}{dt} = -0.013A(t)^2$.

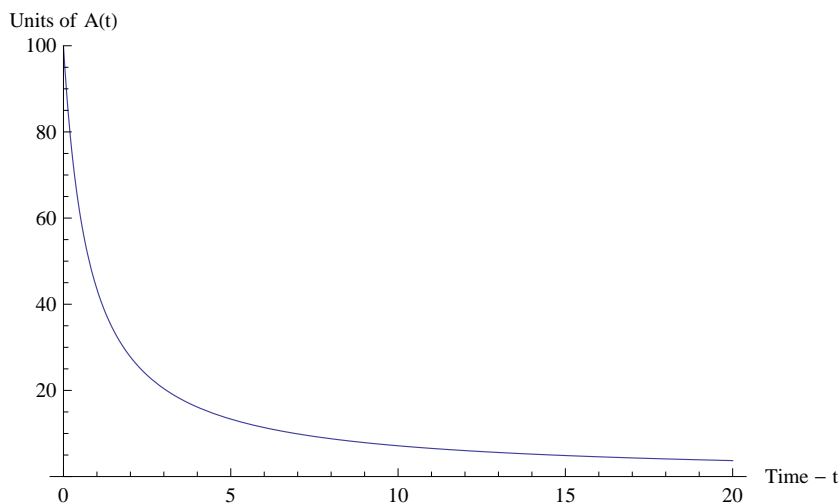


Figure 1. Plot of the solution of differential equation (9) with initial value $A(0) = 100$.

If we were to use, say Mathematica, to solve the original differential equation (9) we would obtain:

$$A(t) = \frac{1000A(0)}{13A(0) \cdot t + 1000} \tag{15}$$

This is the same solution as our by hand solution in (14).

We consider the meaning of our solution (14) in the context of our equation (9). It takes two units (molecules, atoms, moles, etc.) of $A(t)$ to form some product and while this formation is occurring the amount of $A(t)$ present is decreasing at a rate of $0.013A(t)^2$ units of $A(t)$ per unit of time, t . Thus we are constantly losing $A(t)$, but at a slower and slower rate, for the amount of $A(t)$ present is decreasing. A plot of our solution in Figure ?? with, say, $A(0) = 100$ units of $A(t)$ shows this decline in $A(t)$. Incidentally, from its form in (14) we can see that $\lim_{t \rightarrow \infty} A(t) = 0$, another reasonable consequence of our model.

Separation of Variables Could Be Hard Or Impossible

A separation of variables strategy might not be a wise or even doable choice. That does not mean a solution strategy cannot be found with by-hand methods, but it just might not be this strategy. Consider a loan pay back model for a \$100,000 loan in which the interest is 3% annually compounded continuously and goes up 0.05% per year while the amount for the payoff is initially \$4,250 per year

and increases by 1% each year. Convince yourself that a reasonable model for the balance, $b(t)$, on the loan after t years for such a situation is:

$$b'(t) = (.03 + .0005t)b(t) - 4250 * (1 + .01t) \quad \text{with} \quad b(0) = 100000. \quad (16)$$

Can we separate this equation to apply our separation of variables technique? Go ahead and give it a try. Try to explain to yourself or write out in words why separation of variables will not work.

Can you make up a differential equation for which separation of variables technique will work and a similar one for which it does not? No need to worry about any modeling here, just concentrate on the form of the equation.

Take Them as Far As You Can Go

Now for the models (A) - (S) offered at the start of this section pick several which interest you and see if you can separate the variables and then solve in the manner we used above. Go as far as you can and be able to articulate why you could not go any further or why you are confident in your solution which you produced for a solution.

Hint for (M2) - (M6):

$$\frac{1}{y(t)(K - y(t))} = \frac{A}{y(t)} + \frac{B}{K - y(t)}.$$

Can we find values of A and B which make the above equation true? If so, how does this help us to solve the differential equation?