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STUDENT VERSION

Bifurcations

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STATEMENT

Bifurcation analysis is a useful mathematical tool applied in a wide variety of application areas such as neural network models for human sensory systems [3], quantum motion in molecular systems [6], density dependent models of logistic growth in ecology [4], etc. In these situations, solutions to initial value problems are often sensitive to changes in the value of one or more parameter within the model. In particular, it is possible that an equilibrium solution may suddenly change from being asymptotically stable to unstable as parameter values are varied. For example, in quantum mechanics a system of particles may change phase at absolute zero temperature based upon changes in quantum momentum or some other parameter. Whenever quantum momentum is high enough, the system is typically disordered, while at lower momentum values a topologically ordered state of matter may be observed [6]. Unlike thermal phase transitions (*e.g.*, between solid and liquid states) where changes in pressure at a fixed positive temperature may produce mesophases such as liquid crystal, a quantum change of phase occurs suddenly for a single value of one or more parameters. Frequently, these changes are modeled using differential equations for local quantities such as magnetization where a singularity in a derivative indicates the instantaneous change in topological order [2].

In the setting of ordinary differential equations in one real variable, models for various biological systems include a similar behavior. Species of animals where reproductive rates depend highly on food availability at a time prior to when eggs hatch have population dynamics modeled by a delay modification to the logistic equation:

$$\frac{dx}{dt} = rx\left(1 - \frac{x(t-T)}{K}\right),$$

commonly referred to as the Hutchinson-Wright equation [1]. Here, r is the usual intrinsic growth rate, T is a delay parameter, and K is the carrying capacity. It is worth noting that whenever $r > 2$, solutions become chaotic [1]. Moreover, there is a critical value, $T = \pi/(2r)$, at which the equilibrium solution $x(t) = K$ shifts between being unstable and asymptotically stable. Hence, whenever the delay parameter T becomes very large, the population will not tend towards carrying capacity in the long run but rather will decay exponentially.

We investigate first-order differential equations that contain an unknown parameter such as this and identify a methodology for characterizing such a shift in the behavior of equilibrium solutions. Our goal is to understand what happens to the qualitative behavior of solutions to differential equations as we vary the value of this parameter. We will construct an array of phase portraits for different values of the parameter and then summarize the qualitative behavior of solutions in a new graph called a *bifurcation diagram*. The name for this diagram comes from the Latin word *bifurcatus*, meaning forked in two. In this first exercise in bifurcation it will become obvious why we use this terminology.

We begin by examining the differential equation

$$\frac{dx}{dt} = r + x^2\,. \tag{1}$$

1) Consider equation (1) for four distinct values of the parameter r: $r = -4$, $r = -1$, $r = 0$, and $r = 1$. In each of these cases, find all equilibria, determine their stability properties, and create a plot of several solutions to the equation where each one begins with a different initial condition. Summarize and report your results in a structured presentation with diagrams and annotation as well as a good narrative.

We make a slight change in (1) and study the results as we consider the differential equation

$$\frac{dx}{dt} = r - x^2\,. \tag{2}$$

2) Consider equation (2) for four distinct values of the parameter r: $r = -4$, $r = -1$, $r = 0$, and $r = 1$. In each of these cases, find all equilibria, determine their stability properties, and create a plot of several solutions to the equation where each one begins with a different initial condition. Summarize and report your results in a structured presentation with diagrams and annotation as well as a good narrative.

We now increase our power of x, thus making a slight change in (2), and study the results as we consider the differential equation

$$\frac{dx}{dt} = rx - x^3\,. \tag{3}$$

3) Consider equation (3) for four distinct values of the parameter r: $r = 1$, $r = -1$, $r = 0$, and $r = 4$. In each of these cases, find all equilibria, determine their stability properties, and

create a plot of several solutions to the equation where each one begins with a different initial condition. Summarize and report your results in a structured presentation with diagrams and annotation as well as a good narrative.

We make a slight change in (3) and study the results as we consider the differential equation

$$\frac{dx}{dt} = rx + x^3\,. \tag{4}$$

4) Consider equation (4) for four distinct values of the parameter r: $r = 1$, $r = -1$, $r = 0$, and $r = 4$. In each of these cases, find all equilibria, determine their stability properties, and create a plot of several solutions to the equation where each one begins with a different initial condition. Summarize and report your results in a structured presentation with diagrams and annotation as well as a good narrative.

Finally, we make a change in (3) and study the results as we consider the differential equation formed by adding a constant to our right hand side of (3):

$$\frac{dx}{dt} = d + cx - x^3\,. \tag{5}$$

First, note that if $d = 0$ we have the same system as in equation (3) which we have studied already. Thus now consider what happens for different values of d.

5) Considering equation (5) for each of the following values of the parameter d, allow the value of the parameter c to vary as in our previous bifurcation studies and find all equilibria, determine their stability properties, and create a plot of several solutions to the equation where each one begins with a different initial condition.

   First, for $d = 1$, find and classify equilibria for varying values of c. Discuss how the fixed point moves as c moves from $c = -4$ to $c = -1$. What happens when $c = 0$? When $c > 0$? What happens if we go up to and beyond $c = 1.8$ or 1.9? Construct a bifurcation diagram for this value of the parameter "d."

   Now repeat this process for $d = 4$, $d = -1$, and $d = -4$. Summarize and report your results in a structured presentation with diagrams and annotation as well as a good narrative.

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