www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

# STUDENT VERSION
# Ballistics Modeling with a Sponge Dart

Peter Howard
Department of Mathematics
Texas A&M University
College Station TX USA

Jean Marie Linhart
Department of Mathematics
Central Washington University
Ellensburg WA USA

## STATEMENT

Ballistics is the science of projectile motion and impact, phenomena well described by Newtonian mechanics. The number of applications of this type of analysis is staggering, ranging from such mundane issues as automobile accident simulations and optimal golfing to the critical studies of missile defense and space exploration. Somewhat less dramatically, in this project we will use Newtonian mechanics to describe the flight of a sponge dart (a Nerf dart or an off-brand), light enough so that air resistance will play a critical role.

## EXPERIMENTAL DATA

The data for this project was collected by firing sponge darts from a toy gun ($3.99, WalMart—it's not all that easy talking a mathematics department into financing this sort of thing).

The angle of the gun and muzzle height were set and the gun was fastened with duct tape in the correct position to an upright stand for firing. For each dart fired, an observer marked the


**Figure 1.** Left: Sponge dart gun and dart. Right: A plastic protractor used for measuring angles.

spot where the dart initially hit the ground, and a tape measure was used to obtain the horizontal distance to this point from the point where the gun was fired. The horizontal distance was measured four times and those four readings were averaged to make the data point. Likewise, a timer was used to make four measurements of the amount of time it takes a dropped dart to reach the ground, and how long it takes a dart fired vertically to reach the ground. The average of these times was recorded in the dataset included in Table 1.

Using this setup, first, we tabulated a set of measurements for distance traveled (by the projectile) versus angle of inclination of the gun, taking angles of inclination 5, 10, 15, ..., 85 degrees. The darts were fired from a height of 0.18 meters.

| Initial height (m) | Dropped/Fired | Time to reach ground (s) |
|---|---|---|
| 4.06 | dropped | 0.95 |
| 0.39 | fired | 2.13 |

| Angle of inclination (deg) | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
|---|---|---|---|---|---|---|---|---|---|
| Distance traveled (m) | 4.37 | 5.23 | 6.95 | 7.84 | 8.17 | 8.69 | 8.81 | 8.99 | 8.95 |

| Angle of inclination (deg) | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 |
|---|---|---|---|---|---|---|---|---|
| Distance traveled (m) | 8.83 | 8.19 | 7.84 | 7.12 | 6.38 | 5.08 | 3.34 | 2.13 |

**Table 1.** The top table contains distance (meters) and time (seconds) data for a dropped dart and for a dart fired straight up. The second and third table contain angle of inclination in degrees and distance traveled (meters) for the sponge darts fired from a height of 0.18 meters.

These fairly simple measurements will suffice for the assignments in this project.

For most of the assignments, estimation of model parameters will be based on the falling dart and on the dart fired straight up. The data in the table of angles of inclination and distances traveled will only be used to evaluate the models. The final assignment will address incorporating the entire collection of data in parameter estimation and evaluating the resulting model.

For those wishing to replicate the original experiment, two people are needed to fire the gun and observe and record measurements. A ruler or other straight edge can be taped to the gun to facilitate angle measurements. It is possible to use video analysis tools such as [**?**], to analyze videos of the dart's motion. These tools often require having an object of a known length such as a meter stick in the video image.

## ASSIGNMENTS

### Analysis in the absence of air resistance

Ignoring air resistance, Newton's equations of motion for an object under the influence of gravity alone are quite straightforward. Letting $y(t)$ represent the height of the object as a function of time and $x(t)$ represent its horizontal distance traveled as a function of time, we have[1]

$$\frac{d^2}{dt^2}y(t) = -g, \qquad \frac{d^2}{dt^2}x(t) = 0 \tag{1}$$

Your tasks are:

1. Use the time for a dart fired straight up to hit the ground (in Table 1) to determine the initial velocity at which the darts are fired.
2. Solve the two second order differential equations in (1) for $x(t)$ and $y(t)$ and use a computer, e.g. Python or MATLAB, to plot the trajectory ($x$ versus $y$) obtained for angle of inclination $\theta = 35°$. Observe that $x(0)$ and $y(0)$ were determined by the experimental set-up and are known to be 0 and 0.18 m respectively. The remaining two initial conditions $x'(0)$ and $y'(0)$ are given by

$$x'(0) = v_0 \cos\theta, \qquad \text{and} \qquad y'(0) = v_0 \sin\theta,$$

   where $v_0$ is the speed with which the gun fires its darts. Make sure the aspect ratio on this plot is 1.
3. Determine the distance, $d(\theta)$, your object travels as a function ofthe firing angle, $\theta$, and compute the angle that maximizes its distance. You may use a computer to help compute the angle that maximizes the distance traveled by the dart. Your answer should be in degrees and accurate to two decimal places.
4. Plot $d(\theta)$ using a computer, e.g. Python or MATLAB, and compare it with the experimental values $d_{\text{exp}}(\theta)$ given in Table 1. Discuss the discrepancies.

[1] We are also ignoring the fact that the gravitational pull on an object above the earth depends on its height above the earth. We take $g = 9.81\text{m/s}^2$.

5. Compute an error, $E$, for your model based on the sum of squared errors,

$$E = \sqrt{\sum_{k=1}^{17} \left(d_{\text{exp}}(\theta_i) - d(\theta_i)\right)^2}$$

where Table 1 contains data $\theta_i$ and $d_{\text{exp}}(\theta_i)$.

### Linear Air Resistance

A typical first attempt at introducing the effects of air resistance into a physical model is through linear air resistance, using terms of the form $-by'(t)$ and $-bx'(t)$. Adding such terms to (1), we obtain

$$\frac{d^2}{dt^2}y(t) = -g - b\frac{d}{dt}y(t), \qquad \frac{d^2}{dt^2}x(t) = -b\frac{d}{dt}x(t), \tag{2}$$

with the same initial conditions as before. We will now work through steps similar to those in the case of no air resistance, except that as the mathematics becomes more cumbersome, we will begin to fall back more on computer tools, e.g. Python or MATLAB.

Your tasks are:

1. Determine the units on constant $b$. Check that the signs are correct so that the air resistance term always works to bring the velocity closer to zero.
2. Develop analytic solutions to (2).
3. Use your solutions to (2) and the experimental data from Table 1 to determine the value of $b$ for this model.
4. Find a (corrected) initial velocity $v_0$.
5. Plot the trajectory ($y$ versus $x$) for angle $\theta = 35°$ along with your similar trajectory in the absence of air resistance. Make sure the aspect ratio of the plot is 1.
6. Analytically, determine an implicit equation for $d(\theta)$, the distance the dart travels given that it was shot with angle of inclination $\theta$. Use a computer to help you determine the value of $\theta$ that maximizes $d(\theta)$.
7. Plot $d(\theta)$ for this model along with the data, and $d(\theta)$ from the model with no air resistance on the same figure, and again discuss the discrepancies.
8. Compute the error $E$ for this model, and compare it to that from the model with no air resistance.

### Physical Air Resistance

In general, linear air resistance is not an adequate model of projectile motion. In this part of the project, you will work through the steps of the previous cases with a more physical form of air resistance. Unfortunately, analytic solutions for your model will be too cumbersome to derive or work with, so you will have to do most of your analysis with a computer, e.g. Python or MATLAB.

Your tasks are:

1. Use the method of dimensional analysis to determine a more physical form for your air resistance. Give new differential equations with the new physical air resistance term.
2. Check your work. Make sure the air resistance term always works to decrease the velocity. Once again you use an unknown constant $b$ in the air resistance term; what are its units?
3. Use a computer and the experimental data from Table 1 to determine the value of $b$ for your model.
4. Use your new value for $b$ to determine a (further corrected) initial velocity $v_0$.
5. Use Python's `SciPy.integrate.solve_ivp()` or MATLAB's `ode45()` and *event location* to plot a trajectory for angle $\theta = 35°$, and plot it along with the trajectory without air resistance and with the trajectory with linear air resistance at this angle.
6. Plot $d(\theta)$ for this model along with the data and $d(\theta)$ with no air resistance and with linear air resistance. Use a computer to help you determine the value of $\theta$ that maximizes $d(\theta)$ for this model.
7. Compute your error $E$ for this model and compare it with your errors from the model without air resistance and the model with linear air resistance.

## Regression Approach

If we have the same amount of confidence in each data point, we expect to obtain better values for $b$ and $v$ by using more of the data in Table 1 to estimate these parameters.

Your tasks are:

1. Using the values of $b$ and $v_0$ from the physical air resistance model as starting values, carry out a nonlinear least squares regression fit for the function $D(\theta, b, v_0)$ taken to return the horizontal distance traveled by a dart launched with angle of inclination $\theta$, initial velocity $v_0$, and coefficient of air resistance $b$. You will obtain further corrected values for $b$ and $v_0$.
2. Plot your fit for $D(\theta, b, v_0)$ along with your plots of $d(\theta)$ from the physical air resistance model.
3. Compute the error $E$ for the model with these parameter values, and compare to the physical air resistance model.

Wikipedia provides a good general overview of projectile motion, covering many of the ideas covered in this project [1].

## REFERENCES

[1] Wikipedia Contributors. 2018. Projectile Motion. *Wikipedia, The Free Encyclopedia*. Available from `https://en.wikipedia.org/wiki/Projectile_motion`. Last checked 04 September 2018.