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Problem C- Snakes in the Long Run

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Problem: Given that the sex ratio changes as a function of temperature, model the pine snake population dynamics as temperature changes over the course of an incubation period.

Introduction:

Based off of constant lab temperatures, scientists have found a linear model for the sex ratio of pine snakes as a function of incubation temperature \((Sex \text{ Ratio} \approx 0.68 \times T - 0.95)\). We propose that in nature, where the temperature fluctuates throughout the incubation period, the ratio \((R)\) will change as a function of \(\Delta T\) (the change in temperature).

For simplification, we assume that all clutches of eggs will be laid at the ideal hatching temperature, 28°C (so \(T(0) = 28°C\)). Experimental data shows that pine snakes lay their eggs with an initial sex ratio of 1, so when \(\Delta T = 0, R = 1\). The incubation period will also be set as a constant according to ideal conditions, where \(t_h = 64\) days. We also assume that there is no environmental carrying capacity for the snake population.

Based off of the trial in which eggs were incubated at 21°C for 70 days and then raised to 23° for the remainder of the incubation period (a total of 103 days), we assume that the sex ratio of eggs which hatch is dependent on the average temperature for the last third of incubation \((1 - \frac{70}{103}) \approx \frac{1}{3}\). This value will be defined as \(T_3\). Since we propose that the sex ratio changes according to \(\Delta T\), and only \(T_3\) is significant throughout the incubation period; \(R\) will be a function of \(\Delta T_3\).

Using the data from the first trial of Table 4 (Burger) we assume that all pine snake eggs will die at temperatures of 21°C, which is 7° below the “ideal temperature”. For simplification, we assume symmetry, and state that all snake eggs will also die at temperatures of 7° above the ideal temperature (35°C). When all snakes die, we conclude that the ratio goes towards its limits, \(±\infty\).

Analysis:

The basic function for population growth is \(\frac{dP}{dt} = kP\) where \(k\), the growth rate is \((k_{birth} - k_{death})\).

\(k_{birth}\) is a function of the ratio of females \((F)\), the reproduction rate per female \((Q = 8\) eggs/female (Burger)), and the percent hatched \((H)\). \(k_{birth} = F \times 8 \times H\)

\(k_{death}\) is a constant: \(k_{death} = c\)

\(F\), the ratio of females to total population, depends on the ratio \((R)\) of males to females. \((F = \frac{1}{R+1})\)

\(H\), the percent of eggs that hatch at the end of incubation, is a function of \(\Delta T_3\). The data in Table 4 (Burger) can be used to create a cubic line of best fit:

\[
H(\Delta T_3) = -0.7872(\Delta T_3)^3 + 4.9801(\Delta T_3)^2 - 9.9102(\Delta T_3) + 96.388
\]

\(\frac{100}{100}\)
\( R \), the sex ratio, is also a function of \( \Delta T_3 \), which is modeled as a tangential function with asymptotes at \( \Delta T_3 = \pm 7^\circ C \). Using the fact that \( R(0) = 1 \) and testing points from Table 4 (Burger) to scale the function, we arrive at: 
\[
R(\Delta T_3) = 0.319 \tan \left( \frac{\pi \times (\Delta T_3)}{14} \right) + 1.
\]

Given a function \( T(t) \) that measures the environmental temperature vs. time we can compute the average temperature over the last third of incubation: 
\[
T_3 = \frac{3}{t_h} \int_{t_h}^{64} T(t) \, dt
\]
and the more important value, \( \Delta T_3 = \lvert 28 - T_3 \rvert \). Using the assumption of \( t_h = 64 \) days, we simplify to:
\[
\Delta T_3 = \left| 30 - \frac{1}{21} \int_{43}^{64} T(t) \, dt \right|
\]

**Model:**

Given the function \( T(t) \) which graphs temperature against time,
\[
\frac{dp}{dt} = \left[ \left( \frac{b+H(\Delta T_3)}{R(\Delta T_3)+1} \right) - c \right] \times P(t)
\]
Which we simplify as:
\[
\frac{dp}{dt} = \left[ \left( \frac{b+H(\Delta T_3)}{R(\Delta T_3)+1} \right) - c \right] \times P(t)
\]
\[
H(\Delta T_3) = \frac{-0.7872(\Delta T_3)^3 + 4.9801(\Delta T_3)^2 - 9.9102(\Delta T_3) + 96.388}{100}
\]
\[
R(\Delta T_3) = 0.319 \tan \left( \frac{\pi \times (\Delta T_3)}{14} \right) + 1
\]
\[
\Delta T_3 = \left| 28 - \frac{1}{21} \int_{43}^{64} T(t) \, dt \right|
\]

**Conclusion:**

After a series of lab experiments, scientists concluded that the sex ratio of pine snakes increases with temperature. This means that warmer environments result in more male hatchlings that survive incubation. Data also showed that the rate of hatching at the end of incubation is related to temperature, with ideal conditions resting around 28℃. At the far ends of “too hot” and “too cold,” zero eggs will hatch, sending this ratio towards a limit of ±∞. One final consideration is the case in which the hatching rate increased for a clutch of eggs whose temperature was raised during incubation, leading us to conclude that these rates are dependent on the temperature for the final third of incubation.

We propose a population dynamic in which the number of new eggs depends on both the sex ratio and the hatching rate, both functions of a changing temperature. To find the change in population over one hatching season, take the graph of temperature vs. time and use the above equations to calculate

1) \( T_3 \) and \( \Delta T_3 \),
2) \( R(\Delta T_3) \) and \( H(\Delta T_3) \)
3) \( F \)
4) \( k_{birth} \)
5) \( \frac{dp}{dt} \)

For a model of the population over multiple seasons, repeat this set of equations in a loop, where \( P(i + 1) = \frac{dp}{dt} + P(i) \). As average temperature continues to increase, \( \frac{dp}{dt} \) will decrease because the number of females decreases, causing fewer births and a smaller value of \( k_{birth} \). Once \( k_{death} \) outweighs \( k_{birth} \), \( \frac{dp}{dt} \) will become negative and overall population will decrease. A symmetric relationship will occur as average temperature decreases over many seasons. When the temperature reaches either limit of “too hot” or “too cold” \( \frac{dp}{dt} \) will immediately become \( -k_{death} \times P \) and population will drop towards zero.
Sources: