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STUDENT VERSION

Coupled Differential Equations and Heating of a Polycarbonate Block

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The system

An open system where a solid object is being heated up is a standard component of many applications. One such solid that gets used in the design of baby warmers is a polycarbonate sheet [1, 2]. See Figure 1.

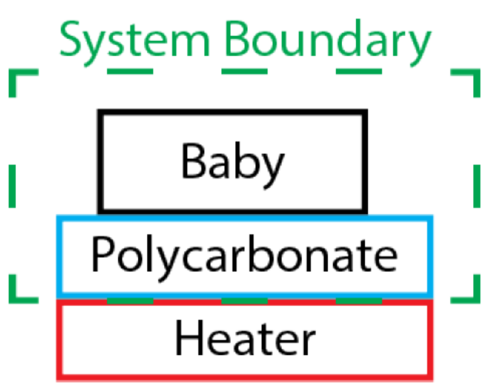

Figure 1. Sketch of Baby warmer using polycarbonate plate.[2]

Here we analyze the case of an open system made up of a polycarbonate block being heat up by a source of temperature $s_{in}(t)$ in presence of a proportional control of the source heat given by $c_1(T_h(t) - T_p(t))$.

The complete system of differential equations is the following,

$$\begin{aligned} T_h'(t) &= s_{in}(t) - c_1(T_h(t) - T_p(t)) \\ T_p'(t) &= c_1(T_h(t) - T_p(t)) - c_2(T_p(t) - T_E). \end{aligned} \tag{1}$$

The first equation says that the heater temperature $T_h(t)$ depends on the source strength $s_{in}(t)$ and on the temperature difference between the heater and the polycarbonate. Here the loss of temperature to the environment has been neglected. The second equation expresses the fact that the temperature of the polycarbonate $T_p(t)$ depends on the heater temperature and on the temperature of the environment, T_E. Here we assume that the constants c_1, c_2 the source term $s_{in}(t)$ are appropriately defined using relevant dimensions.

In control problems in engineering systems, such systems of linear ordinary differential equations, are needed to be represented in terms of simpler cause and effect blocks. The Laplace transform of the differential equations, which connects the input variables to the output variables is calculated. We break the problem down into the following cause and effect reasoning for deciding the input and output of the Laplace transfer function blocks.

1. An input power $s_{in}(t)$ that causes the heater temperature $T_h(t)$ to rise. In the first transfer function block the Laplace transform of $s_{in}(t)$ is the input and the Laplace transform for $T_h(t)$ is the output.
2. The heater temperature $T_h(t)$ causes a rise in the polycarbonate temperature $T_p(t)$ using convection. Thus in the next block the Laplace transform for $T_h(t)$ is the input and the Laplace transform for $T_p(t)$ is the output.
3. Proportional control of the source is represented by the feedback loops that take in $T_p(t)$ and $T_h(t)$ and feed back with amplification factors c_1 and $-c_1$ respectively to the source. In other words the control term $c_1(T_h(t) - T_p(t))$ increases of reduces the effecct of the source term to control the heater temperature $T_h(t)$.

$T_E = 0$ can be assumed to be zero, which is equivalent to working with the shifted temperature scale $T - T_E$. Thus we replace $T_h(t) - T_E$ simply with $T_h(t)$ here. In this new temperature scale, the system of differential equations becomes,

$$T_h'(t) = s_{in}(t) - c_1(T_h(t) - T_p(t)) \tag{2}$$

$$T_p'(t) = c_1(T_h(t) - T_p(t)) - c_2 T_p(t)\,. \tag{3}$$

We describe the connection of this system of linear coupled differential equations with the corresponding block diagram using Laplace transform with cause and effect analysis.

0.1 From the Source to the Heater

Consider the part of the system where the input to the system $s_i n$ gives rise to the heater temperature $T_h(t)$. Here we neglect the polycarbonate completely. The equation of the system is,

$$T_h'(t) = s_{in}(t)\,.$$

We shall consider the general case where $s_{in}(t)$ is a function of time. The Laplace transform of a real function is defined by

$$\mathcal{L}(f(x)) = \int_0^\infty f(x)e^{-sx}dx = \mathcal{F}(s).$$

Thus we take Laplace transforms of both sides,

$$s\mathcal{T}_h(s) - T_h(0) = \mathcal{S}_{in}(s)\,.$$

Assuming $T_h(0) = 0$, i.e. the heater is at the room temperature to start with, the transfer function for this part of the system is,

$$\mathcal{G}_1(s) = \frac{\mathcal{T}_h(s)}{\mathcal{S}_{in}(s)} = \frac{1}{s}$$

When the temperature $T_h(t)$ is fed back into the heat source one gets the resulting differential equation,

$$T_h'(t) = s_{in}(t) - c_1 T_h(t)\,. \tag{4}$$

In the diagram of the Laplace transfer function, this loop combined with the heater block can be represented by a new transfer function block, $\mathcal{G}_1'(s)$, in (6), obtained from the transform of (4) found in (5). Again, we assume $T_h(0) = 0$.

$$s\mathcal{T}_h(s) = \mathcal{S}_{in}(s) - c_1\mathcal{T}_h(s)\,. \tag{5}$$

Now solving for $\mathcal{G}_1'(s) = \frac{\mathcal{S}_{in}(s)}{\mathcal{T}_h(s)}$ we obtain,

$$\mathcal{G}_1'(s) = \frac{1}{s + c_1}\,. \tag{6}$$

0.2 From the Heater to the Polycarbonate

Next let us consider only that part of the system where the heating element temperature, $T_h(t)$ is input to the system and the polycarbonate temperature $T_p(t)$ is the output. This part of the system is modeled by,

$$T_p'(t) = c_1(T_h(t) - T_p(t)) - c_2 T_p(t)\,. \tag{7}$$

After taking the Laplace transform of (7) we write the equation in s-space:

$$s\mathcal{T}_p(s) = c_1(\mathcal{T}_h(s) - \mathcal{T}_p(s)) - c_2\mathcal{T}_p(s)\,.$$

The transfer function for this part of the system is,

$$G_2(s) = \frac{c_1}{s + c_1 + c_2}\,.$$

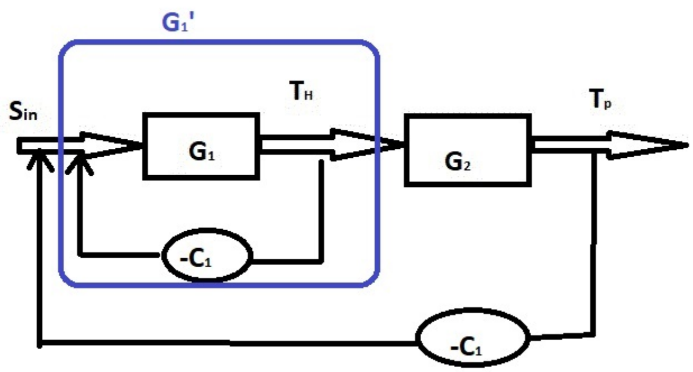

Figure 2. The block diagram representation of the system

0.3 From the Polycarbonate to the Source

The remaining system now consists of input $s_{in}(t)$, the linearly connected blocks, G_1' and G_2 and the feedback loop from $T_p(t)$ to the input. Making up the complete system of differential equations,

$$T_h'(t) = s_{in}(t) - c_1(T_h(t) - T_p(t)), \tag{8}$$

$$T_p'(t) = c_1(T_h(t) - T_p(t)) - c_2 T_p(t). \tag{9}$$

The blocks $\mathcal{G}_1'(s)$ and $\mathcal{G}_2(s)$ can be combined together again as a block with the transfer function,

$$\mathcal{G}_2'(s) = \mathcal{G}_1'(s)\mathcal{G}_2(s) = \frac{c_1}{(s+c_1)(s+c_1+c_2)},$$

and the feedback loop from $T_p(t)$ to $s_{in}(t)$ gives the transfer function for the combined open system,

$$\mathcal{G}_{open}(s) = \frac{\mathcal{G}_2'(s)}{1 - c_1\mathcal{G}_2'(s)}.$$

After substitutions, the transfer function of the system can be written as,

$$\mathcal{G}_{open} = \frac{c_1}{(s+c_1)(s+c_1+c_2) - c_1^2}.$$

Comparing with the eigenvalues

Writing the system in the matrix form,

$$\begin{pmatrix} T_h'(t) \\ T_p'(t) \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 \\ c_1 & -c_1 - c_2 \end{pmatrix} \cdot \begin{pmatrix} T_h(t) \\ T_p(t) \end{pmatrix} + \begin{pmatrix} s_{in}(t) \\ 0 \end{pmatrix}.$$

It can be seen that the characteristic polynomial of the matrix,

$$A = \begin{pmatrix} -c_1 & c_1 \\ c_1 & -c_1 - c_2 \end{pmatrix}$$

is the same as the denominator of $\mathcal{G}_{open}$. Thus the singular values of the transfer function are eigenvalues of the matrix equation, defining the growth rates of the independent solutions.

Activities

Consider our model (1) with $c_2 = 1.6$ and $c_1 = 9$, $s_{in}(t) = 7\left(1 - e^{-0.01t}\right) + 30$ and all initial temperatures at a reasonable room temperature of 20^oC, i.e. $T_h(0) = T_p(0) = T_E = 20$.

1. Verbally describe the situation in this model with these parameters.
2. Solve the model (1) with these parameters using either a numerical or analytic technique (not Laplace transforms and plot on the same axes the three functions, (a) constant body temperature 37^oC, (b) temperature of the heater, $T_h(t)$, and (c) temperature of the polycarbonate block, $T_p(t)$. Explain your plot. See Figure 1.
3. Solve the model (1) with these parameters using Laplace transforms and plot on the same axes the three functions, (a) constant body temperature 37^oC, (b) temperature of the heater, $T_h(t)$, and (c) temperature of the polycarbonate block, $T_p(t)$. Explain your plot. See Figure 1.
4. Compare your results from the previous two solutions in (2) and (3).

Summary

Using transfer functions for analysis of long time behavior of a linear control system has multiple advantages, in terms of flexibility and modularity. However for coupled systems of differential equations a direct connection between the differential equations, the physical causes and effects, and their connection to the Laplace transfer functions may often not be easy to see. Here we have offered an engineering control problem in smaller steps and connected these three ways of analyzing systems.

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