STUDENT VERSION
RIVER CROSSING

STATEMENT

In Figure 1 we see a depiction of a boat crossing a river. We shall model such a crossing using a Free Body Diagram which will enable us to apply Newton's Second Law of Motion. Basically, Newton's Second Law of Motion says that the mass times the acceleration $(m \cdot s''(t))$ of a body is equal to the sum of all external forces which apply to that body. A good Free Body Diagram will enable us to see all these external forces and then build a model (a differential equation model) to describe the position of a body, in this case a boat.

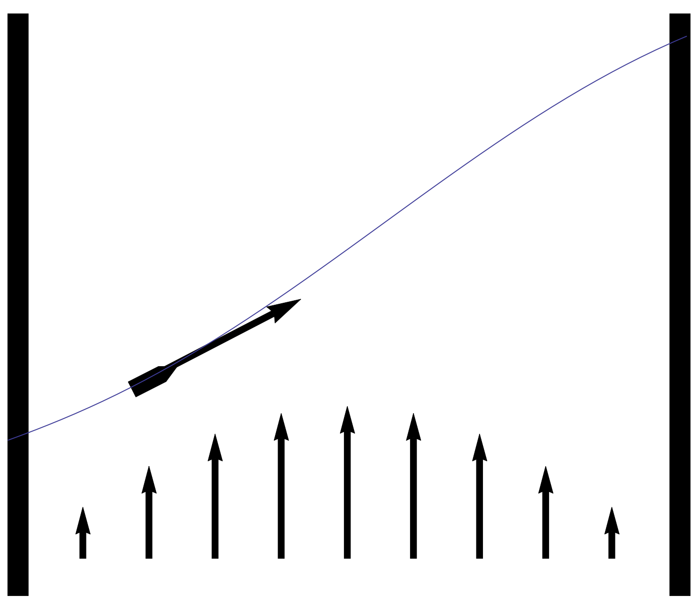

Figure 1. Depiction of river crossing showing upstream river current profile with strength and direction at bottom, resulting curved path of the boat, boat, tangential arrow indicating thrust of the boat's engine, and river banks on each side of the river.

Let us look at the situation. The straight river section is a constant 400 m wide and flowing South to North with maximum velocity or current of 0.9 m/s in the center. The stream velocity is symmetric about the center of the river and decreases to effectively 0 m/s at the bank contact point due to friction.

1) Offer several models of the current velocity profile across the river and defend each one. We are looking for something like $W(x)$ is the velocity in m/s as a function of the distance x m from the West side of the river. Distance is measured perpendicular to the river bank. We will need this in considering how the boat travels across the river, for the velocity of the water in the river will carry the boat along.

Let us agree to identify the variables in question as follows.

- $x(t)$ is the horizontal position in meters from the West bank in a perpendicular direction to the bank of the boat at time t in seconds and thus $0 \leq x(t) \leq 400$ for $t \geq 0$.
- $y(t)$ is the vertical position in meters from the starting coordinate on the West bank in a direction parallel to the shore of the boat at time t in seconds and $y(t) > 0$ signifies the boat is North of its starting position while $y(t) < 0$ signifies the boat is South of its starting position.

Here is some other information which should help in model building.

- The mass of the boat is $m = 225$ kilograms.
- The constant thrust of the boat's engine is $T = 65$ Newtons.
- The coefficient of drag the boat experiences as it travels through water is $c = 25$ N/(m/s).
- The heading of the boat is $\alpha = 10^\circ$ measured off the horizontal West-East line.

Newton's Second Law of Motion applied to a Free Body Diagram

Newton's Second Law of Motion says that for a body of given mass the mass times the acceleration is equal to the sum of the external forces acting on the body. This is true in both the horizontal (x) and vertical (y) components of our boat as it crosses the river. Thus we are looking for something like these two differential equations:

$$\begin{aligned} m\,x''(t) &= \sum_i F_{x\,i} \\ m\,y''(t) &= \sum_i F_{y\,i} \end{aligned} \tag{1}$$

where $F_{x\,i}$ is a force in the x direction acting on the boat and $F_{y\,i}$ is a force in the y direction acting on the boat. We sum over all the forces acting on the boat in each direction.

2) Draw a Free Body Diagram of the boat, i.e. isolate the boat and consider it as a point mass. Then, first list all the forces acting on the boat in each of the coordinate directions, x and y. For example in which direction and in which direction ONLY does the force of the current act on the boat? When does the drag coefficient play a role?

3) Define the magnitude and direction of the forces acting on the free body diagram of the boat and define their functional form in terms of x, y, and t according to the information we are given about the boat crossing.
4) Complete the differential equations of motion, (1), with each force and its appropriate sign and give initial conditions on position and velocity in each direction, x, and y.
5) Solve the differential equations you obtain in activity (4) and plot your solution over sufficient time so the boat gets to the other side of the river. Report out the time for the journey and the vertical coordinate of the landing place on the other side of the river. Explain why your answer is reasonable or not. If not, then change your model, being sure to check that you have all the forces acting on the boat accounted for and your signs and numerical values are correct.
6) Using your base model from activities (4)-(5) and your current velocity profile across the river, $W(x)$, i.e. the North flow rate in m/s at distance x m from the West side of the river, increase (and then decrease) the total flow rate say by 1%, 5%, 10%, 20%. For each of these different flow rates determine the new heading angle, α, the boat person needs to use to land at the same spot as was found in activity (5). Determine a relationship between the percent change in your flow stream and the heading angle, α, and then use it to predict α for a given percent increase in the flow stream you had not computed, all to land at the same spot as was found in activity (5). Hint: How do you know how to increase the TOTAL flow rate by a fixed percent?
7) Holding all else constant, as prescribed before activity (2), just change the total flow rate by various percentages (increase and decrease) and relate the landing position to the percent change, i.e. try to establish a functional relationship between the percent increase in the flow rate and the downstream landing coordinate.
8) Show that if we multiply the vertical velocity profile across the river, $W(x)$, by x and a constant, say β, to keep the TOTAL flow rate the same that the flow rate is faster near the East bank and that if we multiply $W(x)$ by $400-x$ and a constant to keep the TOTAL flow rate the same that the flow rate is faster near the West bank. Do each, solve the resulting differential equations, and then see how the downstream landing coordinate on the East bank changes. Comment on your find. Incidentally, such flow rate changes across a river occur at turns or bends, but can occur in straight flows due to river bottom configurations.

   Hint: In one case you will need to find β such that $\int_0^{400} W(x)\,dx = \beta \int_0^{400} x\,W(x)\,dx$.

Across the river - straight shot

Consider the following model which may have been used in the previous section:

$$
\begin{aligned}
mx''(t) &= T\cos(\alpha) - c\,x'(t)\,, &(2)\\
my''(t) &= T\sin(\alpha) - c\,y'(t) + c\,W\sin\left(\frac{\pi x(t)}{L}\right). &(3)
\end{aligned}
$$

9) Use almost the same numbers used above in model.

- The mass of the boat is $m = 225$ kilograms.
- The constant thrust of the boat's engine is $T = 65$ Newtons.
- The coefficient of drag the boat experiences as it travels through water is $c = 25$ N/(m/s).
- The heading of the boat is α measured off the horizontal West-East line, but is to be determined.
- The maximum speed of the current $W = 0.9$ m/s.

Solve the equation for $x(t)$ in (2), replace $x'(t)$ in the equation for $y(t)$ in (3) and then proceed to determine the heading α which will assure a "straight shot" crossing, i.e., $y(tL) = 0$ at the time of landing, tL, on the opposite shore of the river. Confirm your result.