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STUDENT VERSION
A Matter Of Some Gravity

Kurt Bryan
Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute IN USA

THE PENDULUM

You've been told many times that the gravitational acceleration g at the surface of the earth is roughly $g = 9.8$ meters per second squared. Wouldn't it be nice if you didn't have to take it purely on faith or authority? Why not just measure it yourself with simple equipment, say a ruler and stopwatch? One approach is to drop an object from a given measured height and time how long it takes to fall to the ground. The usual formula $d = \frac{1}{2}gt^2$ (which we won't derive here) for the distance d fallen by an object in a time interval t can be used to solve for $g = 2d/t^2$. The value of d can be measured accurately, but dropping an object from a modest height (say, one meter) results in a pretty short time interval that's hard to measure accurately with simple equipment, so the estimate of g would be crude. Dropping from a larger height to lengthen the time interval brings significant air resistance into the picture.

But the general approach above can work: we need to conduct an experiment whose basic physics can be modeled mathematically (like $d = \frac{1}{2}gt^2$), and into which the unknown value of g enters, but with other parameters that we can readily and accurately measure. One approach is to use a pendulum. For a pendulum of length L, the differential equation that governs the pendulum's motion is, to good approximation

$$\ddot{\theta}(t) + \frac{g}{L}\theta(t) = 0 \tag{1}$$

where $g > 0$ is the local gravitational acceleration and $\theta(t)$ is the angle in radians that the pendulum makes with a vertical line. This model is based the assumption that there is no friction, and that the amplitude of the pendulum's motion is not "too big," say $|\theta(t)| \leq 0.5$ or so. Equation (1) is a linear second order ODE. The usual procedure in an introductory ODE course is to specify values for g and L, as well as initial conditions, and then solve for $\theta(t)$. Once we have the solution we can use it to describe the pendulum's motion, for example, that the motion is oscillatory, and find the period or amplitude.

AN INVERSE PROBLEM

In this case, however, we're going to turn things around: We'll construct a real pendulum and find $\theta(t)$ by observing the pendulum's motion, and from this information deduce the value of g (we'll just measure L with a meter stick). This is an example of an *inverse problem*: we have a differential equation that contains some unknown coefficients, and are given information about the solution to the DE. We then seek to recover the unknown coefficient(s) from this information. The term *parameter estimation* is also often used, and roughly synonymous. The traditional task of solving the ODE for $\theta(t)$ given g and L is called the *forward problem*.

In the present case we'll conduct an experiment in which we deflect the pendulum to an initial angle θ_0 (so $\theta(0) = \theta_0$), let it go with zero initial angular velocity ($\dot{\theta}(0) = 0$), and then measure the period of the motion (we won't measure $\theta(t)$ for all t).

Problem 1: Verify that the solution to (1) with these initial conditions is $\theta(t) = \theta_0 \cos(t\sqrt{g/L})$, and that the period P of the motion is given by

$$P = 2\pi\sqrt{\frac{L}{g}}. \tag{2}$$

Problem 2: Solve equation (2) for g in terms of P and L.

Class Activity: You and your partner have been given a pendulum of some length. Measure it carefully from the end marked with tape to the "middle" of the mass (as best you can estimate). Then measure the period of the pendulum. One person should hold the pendulum firmly at the marked end (it helps to brace your hand against a desk or such); let it swing back and forth at least 10 times, while the other person times the swings. Keep the initial angle under 30 degrees!

Problem 3: Use your measured valued for L and P along with the answer to Problem 2 to estimate g. Compare your estimated for g to that posted in a reliable source for your area of the world (try an internet search for "local gravitational acceleration.")

INVERSE PROBLEMS QUESTIONS

There are several essential questions one usually asks concerning an inverse problem (and many other problems in applied mathematics):

Uniqueness: Is the available data sufficient to uniquely determine the unknown(s) of interest?
Stability: Do small errors in the data result in small errors in the estimate of the unknown(s)?
Reconstruction: How can we reconstruct the unknown(s) from the data?

For this inverse problem you answered the "Reconstruction" and "Uniqueness" problem by finding an explicit formula for g in terms of P (and L). In other inverse problems it's not so easy!

Problem 4: Suppose the goal here was to find both g *and* L. Explain why both of these parameters cannot be found using only P. Could both be found if we knew the actual function $\theta(t)$? Hint: what would happen if we double both g and L?

Now let's take a look at the Stability issue: how will small errors in our measurement of L and P affect our estimate of g? First, recall a basic technique from multivariable calculus: Suppose $z = f(x, y)$ for variables x, y, z and a differentiable function f. Let x_0 and y_0 be some fixed values for x and y, and Δx and Δy be "small" changes in x and y. Then, to good approximation,

$$\Delta z \approx \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y. \tag{3}$$

where $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. Equation (3) is an example of *linearization*.

Problem 5: Let L, P, and g play the roles of x, y, z respectively in (3). Then $g = f(L, P)$ for the function you found in Problem 2. If L_0 is your measured value for L and P_0 the measured value for P then $f(L_0, P_0)$ is the value for g you estimated in Problem 3. We can interpret ΔL and ΔP as the discrepancy between our measured values and the true values, that is, as errors. We can then use (3) to estimate the error Δg in our estimate of g.

a. Make a reasonable "worst-case" estimate for $|\Delta L|$ (you measured it with a meter stick—how close do you think you were?) Do the same for $|\Delta P|$.

b. Use (3) to find a worst-case estimate for $|\Delta g|$. (Hint: ΔL and ΔP may be of the same sign or opposite signs.) Does the local "official" value for g fall within the error interval around your estimated interval?