www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
Developing an Invasive Species Model

Eric Stachura
Department of Mathematics
Kennesaw State University
Marietta, GA USA

STATEMENT

Your goal in this project is to develop a partial differential equation (PDE) model that governs a population (of birds, fish, insects, etc.) that disperses as it migrates. In doing so, you will explore the following:

- the effects of various nonlinear terms on solutions of PDE models;
- the effects of varying initial population profiles;
- the effects of varying diffusion coefficients; and
- Fundamental solutions of differential equations.

All Mathematica notebooks are available in the Supporting Documents where the Modeling Scenario resides.

First model: the Logistic Equation

Consider the logistic equation

$$\frac{dy}{dt} = ky(A-y), \quad k, A > 0,\ y(0) = y_0 \geq 0 \tag{1}$$

Here, $y(t)$ denotes the population size at time t. The explicit solution of (1) is given by

$$y(t) = \frac{y_0 A e^{Akt}}{y_0 e^{Akt} + A - y_0} \tag{2}$$

For more details on the model, see e.g., [1, Section 3.2].

> **Question 1.**
> Play around with the Mathematica Module `Logistic_Module`. Discuss the effects of changing the growth constant k, the carrying capacity A, and the initial population y_0. Compare your analysis with the solution (1). Does this make sense?

Time Evolution of Spatial Profile Curve

Let's now consider a PDE model by thinking of the population being spread out along a narrow strip of land. Suppose we have an initial population distribution $u_0(x)$, which provides the population at position x along the strip of land. The time evolution of the population is a modification of the logistic equation (1)

$$\frac{\partial u}{\partial t} = ku\left(A(x) - u\right), \quad k, A(x) > 0,\ u(x,0) = u_0(x) \geq 0 \tag{3}$$

> **Question 2.**
> How exactly does this generalize (1)? How can we recover it as a special case of (3)?

In (1), the term A was the carrying capacity. The same holds here–except now it depends on the location! In the Mathematica Module `Time_Evolution_of_Spatial_Profile_Module`, we have taken

$$A(x) = 300 + 100\cos\left(\frac{2\pi x}{50}\right)$$

and the growth constant $k = 0.04$.

> **Question 3.**
> Why might the initial population $u_0(x)$ tend towards $A(x)$ as time evolves? Then, see Mathematica module `Time_Evolution_of_Spatial_Profile_Module`.

> **Question 4.**
> Further questions:
>
> 1. Suppose you change the carrying capacity. Would the initial population still evolve towards $A(x)$? Why is this? Verify with some Mathematica demos in the Module `Time_Evolution_of_Spatial_Profile_Module`. PS: For initial populations, we suggest

various cosine curves (e.g., [3+Cos[2*Pi*x/50])

2. What is the effect of taking larger k on the time it takes $u_0(x)$ to approach equilibrium?

3. Investigate instead the model

$$\frac{\partial u}{\partial t} = k(x)u(A-u) \tag{4}$$

with variable growth parameter. Try various cosine curves for $k(x)$. Explain what you see.

Migrating Populations

Next we consider a PDE model where the time rate of change of the population is proportional to its slope:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}, \quad u(x,0) = u_0(x) \tag{5}$$

This model governs migratory population $u(x,t)$ with a parameter c that controls the speed of the migration.

Question 5.
Explore the Mathematica Module `Migrating_Populations_Module`. What happens as you increase the speed c? What happens when you change the initial population distribution?
We suggest trying the following initial population distributions: Probability density function with various means and standard deviations (e.g., $25, 6$ respectively):
`PDF[NormalDistribution[25,6],x]`.
Note: If $p(x)$ is a density function for some characteristic of a population, then

$$\int_a^b p(x)dx = \text{fraction of population for which } a \leq x \leq b$$

If $p(x)$ is a **probability** density function, then

$$\int_a^b p(x)dx = \text{probability that } a \leq x \leq b$$

The probability density of a normal distribution is

$$p(x,\mu,\sigma) = \frac{1}{2\pi\sigma^2}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean of the distribution and σ is the standard deviation. Why might this choice of initial population be reasonable?

We can use Mathematica to solve (5) explicitly. Enter the following code into Mathematica:

```
Module[{u}, pdeMigration=D[u[x,t],t]==-cD[u[x,t],x];
uMigrationSoln[c_,u0_][x_,t_]:= u[x,t] /.
First[DSolve[{pdeMigration, u[x,0]==u0[x]}, u[x,t], {x,t}]];
Row[{"u[x,t] = ", uMigrationSoln[c,u0][x,t]}]]
```

Question 6.
Show that, for any $u_0(t)$, the function $u_0(x - ct)$ satisfies (5).

A solution of the form $u(x,t) = u(x - ct)$ is called a *traveling wave*.

Question 7.
Explore the Mathematica module `Traveling_Wave_Module`. What can you see? How does the sign of c play a role?

Seasonal Migration

To model the periodic nature of certain animals, we can translate the initial population distribution to $u(x,t) = u_0(x - A\sin(ct))$. Then this u satisfies

$$\frac{\partial u}{\partial t} = -Ac\cos(ct)\frac{\partial u}{\partial x}, \quad u(x,0) = u_0(x) \geq 0 \tag{6}$$

Question 8.
Explore the Mathematica module `Seasonal_Migration_Module`. What role does A play in the graph of the solution?

Dispersion

If there is overcrowding in one area, a population may disperse over a wider area. One model for this phenomenon is to say the time evolution of the population is proportional to its concavity. In this case the model becomes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad D > 0 \tag{7}$$

where D is called the dispersion coefficient.

Question 9.
Verify that the fundamental solution of (7)

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{8}$$

actually solves (7). More details on fundamental solutions are provided below.

Question 10.
Now, in the Mathematica module `Dispersion_Module`, explore the effects of the dispersion coefficient on the solution. What happens when D becomes larger?

Why is (8) called a fundamental solution of (7)? Generally, a **fundamental solution** of a PDE is a solution with right hand side given by the Dirac Delta function $\delta(x)$ (see [1, Section 7.9]). So (8) actually solves the equation

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \delta(x)$$

This is beyond the scope of this course, but if you're interested, take a course in PDEs!

Actually, one can also discuss the fundamental solution of an ODE as well! Indeed, given a differential operator L (say, $L = \dfrac{d^2}{dx^2}$), the fundamental solution for L is the solution $u(x)$ of $Lu(x) = \delta(x)$. It is known that

$$\frac{d}{dx}H(x) = \delta(x)$$

where $H(x)$ is the so-called Heaviside function (see [2, Section 6.3]) defined by

$$H(x) = \frac{d}{dx}\max\{x, 0\}, \quad x \neq 0\,.$$

Thus, to find a fundamental solution of $L = \dfrac{d^2}{dx^2}$, it is enough to solve

$$\frac{du}{dx} = H(x) + C, \quad C \text{ constant}\,.$$

The solution of this equation can be found to be

$$u(x) = C_0|x|$$

for some constant C_0. See Figure 1.

Your model

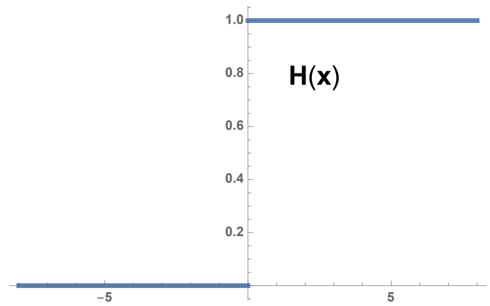

Figure 1. The Heaviside function $H(x)$.

Question 11.
Develop your own model for a population $u(x,t)$ that disperses as it migrates. How many parameters do you have? Can you formulate a fundamental solution of your equation, based on the ideas of the previous section?

REFERENCES

[1] Nagle, R. Kent , Edward B. Saff and Arthur D. Snider. 2018. *Fundamentals of Differential Equations, 7th Edition*. Boston MA: Addison-Wesley.

[2] Boyce, W. E. and R. C. DiPrima. 2009. *Elementary Differential Equations and Boundary Value Problems, 9th Edition*. New York NY: John Wiley & Sons.

POSSIBLE WRITING PROJECT RUBRIC

Each of the following elements are rated along a range of $0 - 4$, depending on the completeness and quality of the work on that element.

1. **Organization:** You must have the following organizational elements in the final paper, each with a clear header.
   - Problem Introduction Section
   - Model Section
   - Mathematical Analysis Section
   - Prediction and Evaluation Section
   - Credits: Give credit to people that assisted you or references you used (if any)
2. **Layout and Formatting:** This is a report, not an assignment, so please write this paper in narrative (paragraph) format, except putting equations and formulas on separate lines. Here are further guidelines:
   - Narratives in paragraphs. Typed 11 or 12 point font.
   - Equations and calculations, etc. are on a separate line and easy to read.
   - Understandability of graphs. (Suggestion: You may take a picture of hand drawn graphs and of graphs on calculators or other software.)
   - Clear identification of graphs in the narratives through labeling of the graphs (e.g., "see Figure 1").
   - Graphs are used effectively in the narratives and explanations. In other words, you refer to the graphs and use the graphs in your discussions.
3. **Introduction:** Must include:
   - Introduction of the problem. (Summarize the problem in your own words.)
   - Discuss how this problem may be similar and different from other applied problems you have encountered.
4. **Model:** Must include:
   - Clear mathematical statement of the model.
   - Definitions of variables.
   - Explanation of the different parts of the model and how they relate to each other.
   - Justification of the model and its parts.

5. **Mathematical Analysis:** Must include:
    - Qualitative Analysis: Discuss what mathematical information the qualitative analysis gives about the solution.
    - Analytical Analysis: Solve and discuss what mathematical information the analytical analysis gives about the solution.
    - Numerical analysis: Solve and discuss what mathematical information the numerical analysis gives about the solution.
    - Explain: Each analysis should include an explanation of the steps taken analyzing the model. (But note you don't have to state or show every single basic algebraic step.)
    - Model verification: Are the results of the model consistent with any data? How? If there is no data, what else can you do to verify if the model results are accurate?
    - Compare/Contrast: Compare and contrast the three techniques (graphical, analytical, numerical). What is similar and different? Did they agree with each other? For this problem, in what ways were they useful or not?
6. **Prediction, Interpretation, and Evaluation:** Must include:
    - Prediction: Make predictions based on the model. See the project directions.
    - Interpretations: Be sure to state what your solution and predictions mean or say in terms of the applied problem. This should be done throughout the write-up as necessary.
    - Evaluation: Critique the interpretations and predictions. Do the interpretations of the solution and predictions make sense for that applied problem? Why or why not?
    - Evaluation: What might your solutions and predictions imply and why is it important for the people, animals, or objects in the applied problem? Does it imply any actions that should be taken in the future, i.e. do you have any recommendations based on your conclusion?