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STUDENT VERSION
It's a Blast (Furnace)!

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STATEMENT

Refined metal is a cornerstone of modern civilization, and industrial furnaces are central to the production of refined metals. These furnaces come in many variations; some are used for the extraction of metal from raw ore (a blast furnace), others to further refine metals from already existing metals (for example, steel from iron), others to reprocess recycled material. There are a wide variety of furnace types and processes.

Whatever type of furnace and process is used, molten material must be contained in a large vessel; see the left panel in Figure 1, a schematic of the situation. One natural question comes to mind: why don't the walls of the furnace melt? The answer is that the vessel, though it may have walls that are themselves made of metal, are lined with a *refractory material*—a material that is resistant to the harsh internal environment and protects the rest of the furnace from the molten charge inside. The refractory material must resist high heat (obviously), but also mechanical wear and chemical attack. Refractory material might be something as simple as brick or other resistant minerals.

Nonetheless, the refractory lining does wear over time. If the lining wears too thin it may result in serious (and dangerous) damage to the walls of the furnace, so it's useful if one can monitor the condition of the refractory lining. This can be done if the furnace is emptied and not in operation, but shutting down the furnace is generally undesirable. It would be better if the lining condition could be determined while the furnace is in operation. This obviously can't be done in any direct fashion. Can we do it from outside the furnace?

One approach is this: if the lining or wall of the furnace begins to thin in a localized area, we might expect that the corresponding portion of the furnace wall on the outside would develop a

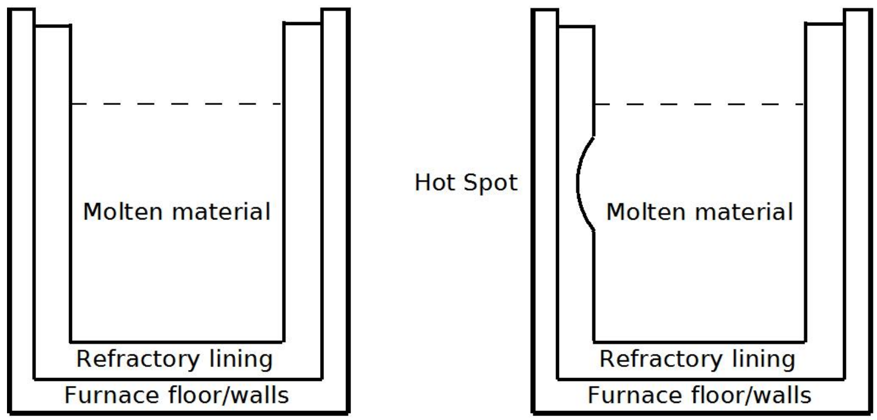

Figure 1. Schematic of an industrial furnace (left), worn refractory lining (right).

"hot spot." This is illustrated in the right panel in Figure 1. And the outside of the furnace is much more easily accessible than the inside! In fact, we can, to a limited extent, get access to the inside of the furnace wall (just not too close to the molten charge inside). One way to do this is to embed temperature sensors (for example, a thermocouple, a passive electrical device that can measure temperature) on or even inside the wall of the furnace; see [4] or [2]. Temperature data can be collected from these sensors and used to estimate the condition of the furnace wall and lining. To do this we need a quantitative procedure for how the temperature of the wall behaves.

Note that the walls are often actively cooled as well, for example, by circulating water, but we won't attempt to model this.

STEADY-STATE HEAT PROPAGATION

Let us start with a simplified model. Our goal here is not a quantitatively precise description of the furnace, but rather a model that can be used for a "proof of concept." In this instance, the concept is that temperature data taken in one region of an object can be used to infer something about the condition of some other part of the object.

To begin, let us assume that the furnace is in a "steady-state" operation, in which temperature and other physical properties of the furnace are changing slowly, or not at all. We will thus assume that the various physical quantities are independent of time. Second, though the furnace is obviously a complex three-dimensional structure, let us focus on a cylindrical section of the wall as illustrated in the left panel of Figure 2. We will assume that the flow of heat energy through the furnace wall is essentially radially outward, and can be modeled by focusing on the flow of heat through this cylindrical section "D" as illustrated in the right panel of Figure 2. Assume the cross-sectional area of this slice D is "A." As you will see below, it doesn't really matter what the precise shape of the cross-section is, nor the precise value of A.

Let $x = 0$ denote the end of D on the external wall, $x = W$ the interface between the furnace

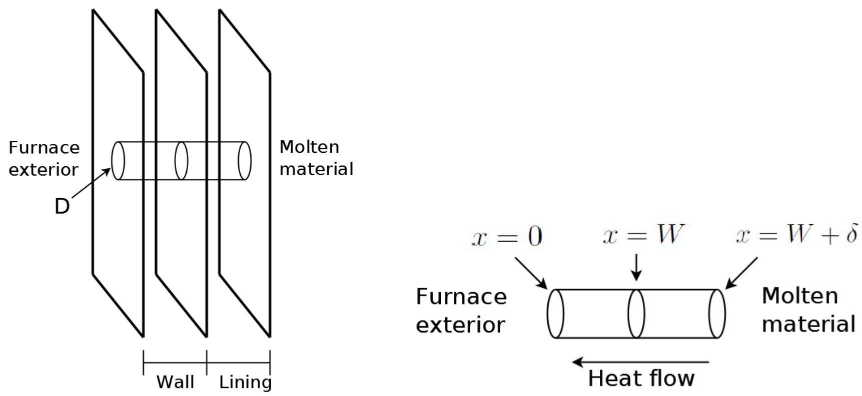

Figure 2. One-dimensional section D of furnace wall (left) and isolated view (right).

wall and the refractory material, and $x = W + \delta$ the other end of D at the interface between the refractory material and molten charge inside the furnace, as illustrated in the right panel of Figure 2. We assume the flow of thermal energy in D is purely horizontal, along the length of D. We use $q(x)$ to denote the rate at which thermal energy is flowing past any given point in D, scaled by A, the cross-sectional area. Thus q has units of energy per unit area per unit time. Equivalently, the amount of heat energy flowing through D at position x is $q(x)A$.

And because we are assuming that the problem is steady-state, the function $q(x)$ must be constant: if $q(x) \neq q(x + \Delta x)$ for some $x, \Delta x$, then the region between x and $x + \Delta x$ in D would have a net gain or loss of thermal energy per unit time, and so would be heating up or cooling down. This is inconsistent with our steady-state assumption. Since $q(x)$ is constant throughout D we have

$$\frac{dq}{dx} = 0 \tag{1}$$

for $0 \leq x \leq W + \delta$.

Let's use $u(x)$ to denote the temperature at each position in D. Normally u would also depend on time, but this problem is steady-state. The conventional model for heat flow is that thermal energy flows in proportion to du/dx, from hot to cold. That is,

$$q(x) = -k(x)\frac{du}{dx} \tag{2}$$

where $k(x) \geq 0$. The function $k(x)$ depends on the material properties of D at position x and is called the *thermal conductivity* of the material. For a material in which physical properties do not depend on position (called a *homogeneous* material) the function $k(x)$ will in fact be constant. Equation (2) is called *Fourier's Law of Heat Conduction*. It's not a fundamental law of nature, but a good approximation of how heat flows through most materials.

THE FORWARD PROBLEM

If we combine (1) and (2) we find

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = 0 \tag{3}$$

must be satisfied by the temperature function $u(x)$. Equation (3) is a second order linear differential equation for $u(x)$, and can be solved by integrating twice. The case in which k is constant is especially easy, for (3) becomes just $u''(x) = 0$.

Problem 1: Integrate $u''(x) = 0$ twice to show that u must be a linear function of the form $u(x) = c_1 x + c_0$.

In our case we will assume that the thermal conductivity is a constant k_1 for $0 < x \leq W$ (the wall of the furnace) and a different constant k_2 for $W < x < W + \delta$ (the refractory lining.) If the wall of the furnace is made of steel, a typical value for k_1 in this region would be $k_1 \approx 25$ watts per meter per degree Celsius. For the refractory lining a value of $k_2 \approx 0.5$ is more in the ballpark (typical of firebrick). These values can vary with temperature, but we won't model this. A typical value for W is $W = 0.15$ meters. The value for δ can vary, but we'll use $\delta = 0.25$ meters for now. See references [3, 1] for typical ranges of these parameter values.

Problem 1 shows that the temperature is a linear function in each region $0 < x < W$ and $W < x < W + \delta$, but they don't have to be the same function! That is, $u(x)$ must be of the form

$$u(x) = \begin{cases} c_1 x + c_0, & 0 < x \leq W \\ c_2 x + c_3, & W < x < W + \delta \end{cases} \tag{4}$$

for some choice of c_0, c_1, c_2, c_3. However, the value of these constants is constrained by some physics.

Problem 2:

a. Temperature varies continuously; informally, an abrupt jump in u would result in arbitrarily large values for u', and so (from (2)) arbitrarily large heat flux, which isn't physically reasonable. As such the solution (4) should be continuous everywhere, in particular, through W. Show that this forces $c_1 W + c_0 = c_2 W + c_3$. Hint: consider the limit of $u(x)$ as x approaches W from the left and from the right.

b. Equation (1) shows that the heat flux $q(x)$ in (2) must be constant in our situation. This is true even through $x = W$ (if more heat was flowing into one side of $x = W$ than was flowing out of the other side, the problem couldn't be steady-state—the region near $x = W$ would be changing temperature.)

   Compute $q(x)$ from (2) and show that we must have $k_1 c_1 = k_2 c_2$. Hint: consider the limit of $q(x)$ as x approaches W from the left and from the right.

c. With $k_1 = 25$, $k_2 = 0.5$, and $W = 0.15$ use parts (a) and (b) to show that

$$c_1 = 0.1361(c_0 - c_3) \text{ and } c_2 = 6.803(c_0 - c_3)$$

(rounding to 4 significant figures). Thus the solution $u(x)$ in (4) can be written as

$$u(x) = \begin{cases} 0.1361(c_0 - c_3)x + c_0, & 0 < x \le 0.15 \\ 6.803(c_0 - c_3)x + c_3, & 0.15 < x < 0.15 + \delta \end{cases} \tag{5}$$

for some constants c_0 and c_3. This function is continuous through $x = 0.15$, as is the heat flux $q(0.15)$.

The function $u(x)$ in (5) describes the temperature of the furnace wall, but still contains two constants c_0 and c_3. These must be found from two appropriate boundary conditions. A furnace typically operates at a controlled, fixed internal temperature, so at the inner wall ($x = W + \delta$) the temperature may be considered known. Let us suppose that this temperature is 1600 degrees Celsius (typical, see [3]). This is one boundary condition.

The second boundary condition comes from the physics at $x = 0$, the outer wall. This stems from the loss of heat through the walls of the furnace to the ambient environment. Let's take a simple model in which the wall of the furnace loses heat in a "Newton cooling" fashion: the heat flux out of the wall (which is proportional to $-u'(0)$) is proportional to the difference between the wall and ambient temperature. Suppose the ambient temperature is $T_A = 30$ degrees Celsius; the Newton-cooling assumption leads to a boundary condition of the form

$$\frac{du}{dx}(0) = k_0(u(0) - T_A) \tag{6}$$

for some positive constant k_0. We will take $k_0 = 0.7$ as a start.

With given boundary conditions, as well as specified values for W, δ, and the thermal conductivity in each region, the problem of determining the temperature profile $u(x)$ is called the *forward problem* or the *direct problem*. It's the traditional thing to do with a differential equation—find the solution! That's what Problem 3 below is concerned with.

Problem 3: Use (5) to show that the condition $u(W+\delta) = 1600$ (with $W = 0.15, \delta = 0.25$) implies that $2.721c_0 - 1.721c_3 = 1600$. Also use (5) and (6) to show that we need $0.136(c_0 - c_3) = 0.7c_0 - 21$. Solve these two equations to find c_0 and c_3, and use this to write $u(x)$ out explicitly.

Then plot the temperature profile $u(x)$ for $0 < x < W + \delta$. Is the plot consistent with $u''(x) = 0$ in each region $0 < x < W$ and $W < x < W + \delta$? Is it continuous through $x = W$? Hint: the temperature of the outer wall should be about 189.3 degrees Celsius.

AN INVERSE PROBLEM

Problems 1 to 3 above show that if we are given all essential parameters, namely W and δ (to specify the geometry), k_1 and k_2 (material parameters) and the boundary data, we can compute the temperature $u(x)$ at any point in the wall. In our case, however, we are interested in the value of δ, the thickness of the refractory lining, which we do not know and cannot measure directly. We have to infer δ from data concerning $u(x)$, hence the name "inverse problem."

Let's suppose that in addition to knowing the Newton-cooling boundary condition at $x = 0$ and the internal temperature at $x = W + \delta$ (even if δ itself is not known), we have an additional piece of data: the temperature $u(0)$, the temperature of the outer wall. If we now treat δ as unknown but use the additional piece of information that $u(0) = u_0$ (here u_0 is the measured temperature datum) can we estimate δ? This is an example of an *inverse problem* in which one uses information about the solution to a DE in order to estimate some unknown parameter that appears in the differential equation.

For an inverse problem (and in fact, many types of mathematical problems) the essential questions of interest are:

- (Uniqueness) Can the unknown of interest be uniquely identified from the data at hand? If not, what additional data or assumptions would make that possible?
- (Reconstruction) Find a constructive procedure for estimating the unknown.
- (Stability) How sensitive is any estimate of the unknown to noise in the data or modeling assumptions?

Problem 4: The goal here is to treat δ as unknown and infer it from an additional measurement of u. We'll assume that we know that $W = 0.15$, $k_1 = 25$, $k_2 = 0.5$ as before, as well as $k_0 = 0.7$ and $T_A = 30$. We'll also suppose that we measure $u(0) = 200$.

a. Note that the solution (5) you obtained in Problem 2 with constants c_0 and c_3 (prior to implementing the boundary conditions at the ends) is still correct. Show that the "measurement" $u(0) = 200$ forces $c_0 = 200$.

b. Use the boundary condition (6) to show that $0.136(c_0 - c_3) = 0.7c_0 - 21$ (same as in Problem 3) and that as a result we have $c_3 = -674.65$. Write out $u(x)$ explicitly and plot it in the range $0 < x < 0.5$.

c. It is still true that $u(W + \delta) = 1600$—why? Use this to find δ. Hint: Just set $u(x) = 1600$ and solve for x—the is where the interface with the molten material is, so you can deduce δ.

Problem 5: Repeat Problem 4 but now suppose $u(0) = u_0$ where u_0 is unspecified. Find an expression for δ in terms of u_0. Does knowledge of u_0 always allow us to find δ?

Problem 6: Based on your answer to Problem 5, can δ always be identified from the given data? (The "uniqueness" question above). What is the reconstruction formula?

Problem 7: Suppose that in Problem 4 we erroneously measure $u(0) = 205$ (instead of $u(0) = 200$). How far off is our estimate of δ? What if $u(0) = 195$? This gets at the Stability issue above.

Problem 8: Suppose that the minimum safe value for the lining thickness is $\delta = 0.15$ (all other parameters as in Problem 3). What is the maximum value for $u(0)$ we can allow before shutting down the furnace for repair?

CONCLUSION

Of course, a real furnace is three-dimensional, the heat flow is time-dependent, and many facets of the operation have not been modeled (actively cooled walls as noted, and the precise boundary condition on the outer face are but two examples). All of this would have to be accounted for if one wanted to actually use these ideas for furnace operation. But, as noted, a simplified model as presented can still give valuable intuition, by illuminating what variables might be most important, and whether what we seek to do can be done at all.

For a version of this problem in which the full time-dependent heat equation is used, see [5] (work done by undergraduates.)

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