STUDENT VERSION
Using Phase Portraits to Analyze Relationship Dynamics

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STATEMENT

We start with a synopsis of the problem as stated in [3]. Romeo and Juliet are in love; however, Juliet is a volatile lover. The more Romeo expresses his infatuation for Juliet, the more she runs away. Interestingly enough, when Romeo is demoralized by Juliet's lack of emotion, Juliet is then suddenly attracted to Romeo. Romeo, in comparison, desires Juliet when she shows interest in him and becomes disinterested when Juliet no longer shows him attention. Let

$$R(t) = \text{Romeo's love/hate for Juliet at time t}$$

$$J(t) = \text{Juliet's love/hate for Romeo at time t}$$

Note that positive values for R and J signify love while negative values signify hate. This behavior of the relationship between Romeo and Juliet can be express as the following system of differential equations:

$$\begin{aligned} \frac{dR}{dt} &= aJ \\ \frac{dJ}{dt} &= -bR \end{aligned} \tag{1}$$

where a and b are both positive real numbers.

Note: You may use the MATLAB codes found in the Appendix as a template for the code needed for some of the problems in this project. For more on using Matlab to numerically solve differential equations, see [1].

Exploration of the model

1. Why do the right hand sides of the equations have different signs?

2. With $a = 1$, $b = 1$, and initial conditions $R(0) = 2$ and $J(0) = 0$, plot the time evolutions of R and J on the same set of axes. Explain why your graphs match the behavior of Romeo and Juliet's relationship as described in the statement of the problem.
3. With $a = 1$ and $b = 1$, sketch the phase portrait of system (1). Use your sketch to conclude that the behavior of Romeo and Juliet's relationship does not depend on how much love or hate Romeo initially has for Juliet.
4. What name would you give to Romeo and Juliet's relationship?
5. With different positive values for a and b, repeat parts (b) and (c). How do your results differ from the results you obtained from (b) and (c) above. Interpret your results in the context of Romeo and Juliet's relationship.

Modifying the Model

Assume that the right hand sides are now both positive i.e

$$\begin{aligned} \frac{dR}{dt} &= aJ \\ \frac{dJ}{dt} &= bR \end{aligned} \tag{2}$$

where a and b are both positive real numbers,

1. Describe the behavior of Romeo and Juliet's relationship captured by this system of equations.
2. With $a = 1$ and $b = 1$, sketch the phase portrait of R and J and explain why your sketch matches the behavior of Romeo and Juliet's relationship you described in Question 1.
3. Come up with three different scenarios of initial feelings and predict how Romeo and Juliet's relationship will eventually turn out? (Hint: Use your phase portrait).
4. What name would you give to this type of Romeo and Juliet's relationship ?

Extending the Model

Suppose Romeo's love for Juliet depends entirely on Juliet's feelings in the following manner: the more Juliet loves Romeo, the more Romeo loves Juliet; and the more Juliet hates Romeo, the more Romeo hates Juliet. Also suppose Juliet's love for Romeo depends on both his feelings towards her and her own feelings; and that the more Romeo loves Juliet, the less interested Juliet is.

1. Write the system of equation that captures this scenario.
2. How similar is your system to (1)? Use only one parameter in your system to regulate how similar the phase portraits of your system are to rhose in (1). Demonstrate this with two phase portraits.

Generalizing the Model

Let us now generalize the system as follows

$$\begin{aligned} \frac{dR}{dt} &= aR + bJ \\ \frac{dJ}{dt} &= cR + dJ \end{aligned} \tag{3}$$

1. With $a = d = -1$ and $b = c = 0.5$, we get a scenario where both Romeo and Juliet are both inclined to go in the opposite of direction of what their hearts want, but are open to being influenced by the other, and thus only show their feelings when the other person shows theirs. Strogatz in [3] terms this scenario "cautious lovers". Sketch the phase portrait of system (3), and use it to predict what will eventually happen to the relationship of these cautious lovers?
2. If $a = d = -0.5$ and $b = c = 1$, Romeo and Juliet are still cautious, but not as much as in (a). Sketch the phase portrait of system (3), and use it to predict what will eventually happen to the relationship of these somewhat cautious lovers?
3. Now Let $a = -d = 1$ and $b = -c = 1$. Sketch the phase portrait of system (3). What can you conclude about Romeo and Juliet's relationship?

Appendix

We present here MATLAB code for solving portions of this modeling activity.

MATLAB code for solving a 2D system

```
% Let Romeo's love for Juliet at time t be R(t)
% Let Juliet's love for Romeo at time t be J(t)

% Problem 1
% R' = aJ  and  J'=-bR
a=1; b=1; c=11; d=1;

tspan =[0 20]; %domain under consideration
ic=[2 0]; % initial condition
dxdt = @(t,x) [a*x(1) + b*x(2); c*x(1)+d*x(2)];
% R is x(1); J is x(2)
[t, x] = ode45(dxdt, tspan, ic);

plot(t,x(:,1),'b',t,x(:,2),'g')
xlabel('t', 'FontSize', 20)
ylabel( 'R(t), J(t)', 'FontSize', 20)
legend ({'R(t)', 'J(t)'}, 'FontSize', 18)
axis([0 20 -2.5 2.5 ])
grid on
```

MATLAB code sketching the phase portrait of a 2D system

```
% IMPORTANT - vectorfield.m must be in same directory.

clear;figure;

a=-0.5; b=1; c=1; d=-0.5;
dxdt = @(t,x) [a*x(1) + b*x(2); c*x(1)+d*x(2)];
% R is x(1); J is x(2)


vectorfield(dxdt,-3:.25:3,-3:.25:3)
     hold on
     for x0=-3:1.5:3
```

```
        for y0=-3:1.5:3
            [ts,xs] = ode45(dxdt,[0 5],[x0 y0]);
            plot(xs(:,1),xs(:,2))
        end
    end
    for x0=-3:1.5:3
        for y0=-3:1.5:3
            [ts,xs] = ode45(dxdt,[0 -5],[x0 y0]);
            plot(xs(:,1),xs(:,2))
        end
    end


axis([-3 3 -3 3])
set(gca,'XTick',-3:1:3,'FontSize',15)
set(gca,'YTick',-3:1:3,'FontSize',15)
xlabel('R(t)','FontSize',15)
ylabel('J(t)','FontSize',15)
hold off
```

MATLAB code sketching vector fields

```
% Save M-file as vectorfield.m. Do NOT run this function file.
% Copyright Springer 2014. Stephen Lynch.

function vectorfield(deqns,xval,yval,t)
if nargin==3;
    t=0;
end
m=length(xval);
n=length(yval);
x1=zeros(n,m);
y1=zeros(n,m);
for a=1:m
  for b=1:n
    pts = feval(deqns,t,[xval(a);yval(b)]);
    x1(b,a) = pts(1);
    y1(b,a) = pts(2);
  end
end
```

```
arrow=sqrt(x1.^2+y1.^2);
quiver(xval,yval,x1./arrow,y1./arrow,.5,'r');
axis tight;

% End of vectorfield.m.
```

REFERENCES

[1] Lynch, Stephen. 2014. *Dynamical Systems with Applications using MATLAB – 2nd Edition*. New York: Springer.

[2] Strogatz, Steven H. 1988. Love Affairs and Differential Equations. *Mathematics Magazine*. 61(1): 35.

[3] Strogatz, Steven H. 1994. *Non-Linear Dynamics and Chaos*. New York: Perseus Books Group. 1994.