www.simiode.org
SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION

Traveling wave analysis of a porous medium model

Eric Stachura
Department of Mathematics
Kennesaw State University
Marietta GA USA

STATEMENT

> NOTE: There is an Introduction to the Problem file in the same folder (`https://www.simiode.org/resources/629`) as this Modeling Scenario. Further there is a video of the process referred to in this Introduction file called drip.mov which can also be found in the same folder under Supporting Docs.

The following model for the flow of water under gravity through a homogenous, isotropic, porous medium is developed in [4] and given by

$$\frac{\partial u}{\partial t} = \frac{\partial u^n}{\partial x} + \frac{\partial^2 u^m}{\partial x^2}, \quad n, m \geq 1,\ x \in \mathbb{R},\ t > 0 \tag{1}$$

where $u(x,t)$ is the volumetric moisture content of the porous medium.

Your task is to analyze the model (1) by developing analytical solutions. Consider the case when $n = 3, m = 2$; this case is suggested by empirical data [1]. Notice this is a nonlinear partial differential equation! A typical approach to solving such problems is by seeking *traveling wave solutions*; that is, seeking solutions of the form

$$u(x,t) = u(x - ct)$$

which represents a wave traveling with speed c. Traveling waves advance in a particular direction and retain a fixed shape. They also retain a constant velocity throughout its propagation; see Figure 1 for an example of such a traveling wave. Notice the wave advances to the right as time progresses and keeps its shape fixed.

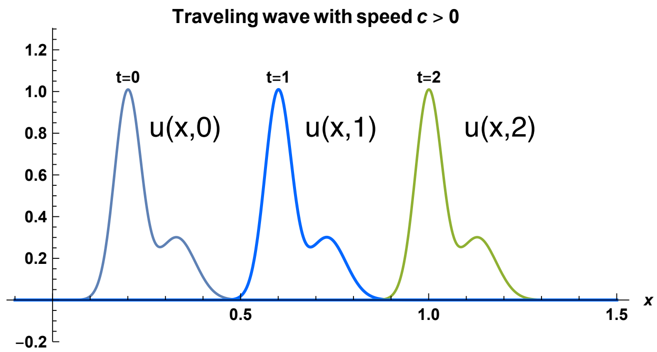

Figure 1. Example traveling wave profile with speed $c > 0$.

Question 1.
Let $z = x - ct$. Assuming a traveling wave solution $u(z)$ as above, what *ordinary* differential equation arises from such an approach?

Question 2.
Now that you have your ODE model, introduce the new independent variable

$$Y = \tanh(\mu(x - ct))$$

where μ is some constant you will determine. Why hyperbolic tangent? Good question. There is a method, called the "tanh method", which assumes a traveling wave solution can be expressed in terms of tangents or hyperbolic tangents [3]. The Appendix at the end provides some information about the tanh function if you have not seen it before.

(i) How do the derivatives $\dfrac{d}{dz}$ and $\dfrac{d^2}{dz^2}$ transform under this change of variables? That is, for a function $v(z)$, how do $\dfrac{dv}{dz}$ and $\dfrac{d^2v}{dz^2}$ transform to $\dfrac{dv}{dY}$ and $\dfrac{d^2v}{dY^2}$ under this change of variables? HINT: Chain rule.

(ii) What is the resulting equation for $u(Y)$?

Question 3.
With your tanh method, an idea is to seek power series solutions. This is because the resulting polynomials tend to be easy to work with (much easier than a nonlinear PDE!). For a reminder of power series solutions, see e.g. [2, Chapter 8]. We seek a solution of the form

$$u(x,t) = S(Y) = \sum_{k=0}^{p} a_k Y^k + \sum_{\ell=1}^{q} b_\ell Y^{-\ell}$$

After balancing various terms (no need to do this–if you're interested, see [4]), the solution is forced to be of the form

$$u(x,t) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1} \tag{2}$$

Now, find a system of equations for the unknowns μ, a_0, a_1 and b_1. HINT: Plug (2) into the solution you obtained in Question 2.

Question 4.
Solve the system of equations you obtained in Question 3. HINT: your solution depends on the sign of c (which dictates the direction of travel for the traveling wave). You should use computer software (e.g., Matlab, Mathematica, ...) to help with this task.

Question 5.
Show then that the linearly independent solutions of (1) are given by

$$\begin{aligned} u_1(x,t) &= -\sqrt{c}\tan(0.5\sqrt{c}(x-ct)), \quad 0.5\sqrt{c}(x-ct) \neq \text{ an odd multiple of } \pi \\ u_2(x,t) &= \sqrt{c}\cot(0.5\sqrt{c}(x-ct)), \quad\quad 0.5\sqrt{c}(x-ct) \neq \text{ a multiple of } \pi \end{aligned} \tag{3}$$

if $c > 0$ and if $c < 0$ then

$$\begin{aligned} u_3(x,t) &= \sqrt{-c}\tanh(0.5\sqrt{-c}(x-ct)) \\ u_4(x,t) &= \sqrt{-c}\coth(0.5\sqrt{-c}(x-ct)), \quad x - ct \neq 0 \end{aligned} \tag{4}$$

Question 6.
Let's analyze your results now. The solutions you constructed $u_j(x,t)$ for $j = 1, 2, 3, 4$ correspond to moisture content in a porous medium (remember this is what the model is for!). This

should be a positive quantity (how could you have negative moisture content?!?). For one of your solutions $u_j(x,t)$, find where it is positive.

Question 7.

Consider $u_1(x,t)$ when $c=1$. From physical considerations, the seepage velocity of the water q satisfies the continuity equation

$$\frac{\partial u_1}{\partial t} + \frac{\partial q}{\partial x} = 0$$

The seepage velocity here corresponds to the ratio of the volume flow rate to the average area of voids in the porous medium. See Figure .

Using this equation as well as your calculation of $u_1(x,t)$, derive an expression for $q(x,t)$. For this to be a physically relevant quantity, we expect $q(x,t)$ to be continuous. Under what conditions on x and t will this be true?

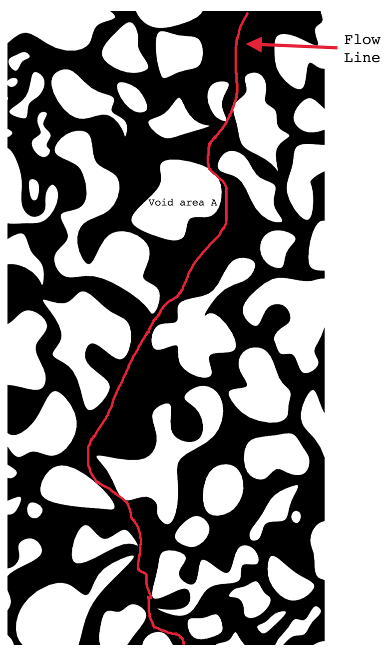

Figure 2. Flow of water in a porous medium. The seepage velocity is the ratio of volume flow rate to the average void area.

Question 8.
Note that changing the values of m and n means your model is changing, so you may be interested in different things. For example, the case $m = 4, n = 3$ models the flow of a thin viscous sheet on an inclined bed. Also, the case $n = 0$ and $m \neq 0$ is called the *porous medium equation*, and models water movement along a horizontal column of the medium.

Think about what might change in your analysis if you took different values of m and n. Could you perform a similar traveling wave analysis? What other types of equations do you think you could apply this method to?

Appendix

One way of defining the hyperbolic tangent is

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Hyperbolic tangent satisfies $\tanh(-x) = -\tanh(x)$, and the following trigonometric identity

$$\operatorname{sech}^2(x) + \tanh^2(x) = 1.$$

where we can define $\operatorname{sech}(x)$ by

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}.$$

Furthermore, when differentiating the hyperbolic tangent, we get

$$\frac{d}{dx}\tanh(x) = 1 - \tanh^2(x) = \operatorname{sech}^2(x).$$

Plots of $\tanh(x)$ and $\operatorname{sech}(x)$ can be found in Figure 3:

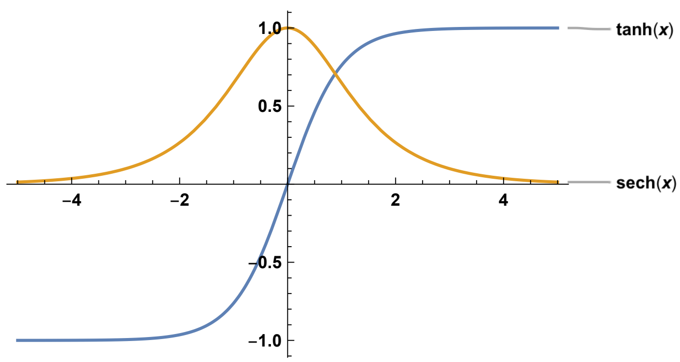

Figure 3. Plots of two hyperbolic trigonometric functions.

Additionally, with complex arguments, the following identities hold

$$\tanh(x) = -i\tan(ix) \quad \text{and} \quad \coth(x) = i\cot(ix).$$

REFERENCES

[1] Bear, J. 1972. *Dynamics of Fluids in Porous Media*. New York: Elsevier.

[2] Nagle, R. Kent, Edward B. Saff, and Arthur D. Snider. 2018. *Fundamentals of Differential Equations, 7th edition*. Boston MA: Addison-Wesley.

[3] Malfliet, W. and W. Hereman. 1996. The Tanh method: I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta*. 54: 563–568.

[4] Vanaja, V. 2009. Exact solutions of a nonlinear diffusion-convection equation. *Physica Scripta*. 80: 045402.

POSSIBLE RUBRIC/ GUIDE FOR STUDENTS

Each of the following elements are rated along a range of $0 - 4$, depending on the completeness and quality of the work on that element.

1. **Organization:** You must have the following organizational elements in the final paper, each with a clear header.
   - Problem Introduction Section
   - Model Section
   - Mathematical Analysis Section
   - Prediction and Evaluation Section
   - Credits: Give credit to people that assisted you or references you used (if any)
2. **Layout and Formatting:** This is a report, not an assignment, so please write this paper in narrative (paragraph) format, except putting equations and formulas on separate lines. Here are further guidelines:
   - Narratives in paragraphs. Typed 11 or 12 point font.
   - Equations and calculations, etc. are on a separate line and easy to read.
   - Understandability of graphs. (Suggestion: You may take a picture of hand drawn graphs and of graphs on calculators or other software.)
   - Clear identification of graphs in the narratives through labeling of the graphs (e.g., "see Figure 1").
   - Graphs are used effectively in the narratives and explanations. In other words, you refer to the graphs and use the graphs in your discussions.
3. **Introduction:** Must include:
   - Introduction of the problem. (Summarize the problem in your own words.)
   - Discuss how this problem may be similar and different from other applied problems you have encountered.
4. **Model:** Must include:
   - Clear mathematical statement of the model.
   - Definitions of variables.
   - Explanation of the different parts of the model and how they relate to each other.
   - Justification of the model and its parts.

5. **Mathematical Analysis:** Must include:
    - Qualitative Analysis: Discuss what mathematical information the qualitative analysis gives about the solution.
    - Analytical Analysis: Solve and discuss what mathematical information the analytical analysis gives about the solution.
    - Numerical analysis: Solve and discuss what mathematical information the numerical analysis gives about the solution.
    - Explain: Each analysis should include an explanation of the steps taken analyzing the model. (But note you don't have to state or show every single basic algebraic step.)
    - Model verification: Are the results of the model consistent with any data? How? If there is no data, what else can you do to verify if the model results are accurate?
    - Compare/Contrast: Compare and contrast the three techniques (graphical, analytical, numerical). What is similar and different? Did they agree with each other? For this problem, in what ways were they useful or not?
6. **Prediction, Interpretation, and Evaluation:** Must include:
    - Prediction: Make predictions based on the model. See the project directions.
    - Interpretations: Be sure to state what your solution and predictions mean or say in terms of the applied problem. This should be done throughout the write-up as necessary.
    - Evaluation: Critique the interpretations and predictions. Do the interpretations of the solution and predictions make sense for that applied problem? Why or why not?
    - Evaluation: What might your solutions and predictions imply and why is it important for the people, animals, or objects in the applied problem? Does it imply any actions that should be taken in the future, i.e. do you have any recommendations based on your conclusion?