A Porous Medium Model

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Problem

We will derive a model for the flow of water under gravity through a homogeneous, isotropic, porous medium.

Definitions

1. Homogeneous: The material looks the same at every location. It has uniform properties throughout; no clumps of other material.

2. Isotropic: The material looks the same in all directions. That is, the material properties do not depend on the direction.

Porous Medium

Intuitively, such a medium has “pores”, or voids, which are typically filled with a fluid. Examples include rocks, soil, biological tissues, etc. You can imagine something as in the following figure:

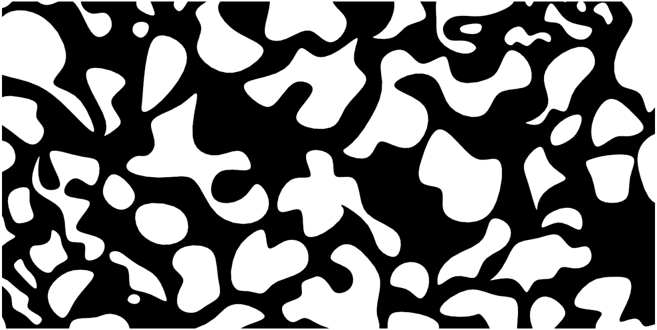

Figure: A porous medium.

The Model

Let us define:

- $u(x,t)$ to be the volume of water per unit volume of the porous media;
- $q(x,t)$ to be the volume of water flowing across a unit area per unit time

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If the density of water is assumed to be constant, which we will assume, then the *moisture content* $u(x,t)$ and the *seepage velocity* $q(x,t)$ of the water are governed by the continuity equation

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

The Model

as well as Darcy's Law

$$q(x,t) = -K(u)\frac{\partial G}{\partial x} \tag{2}$$

where G is some potential function and $K(u)$ is called the *hydraulic conductivity*.

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If chemical and thermal effects are ignored, then for unsaturated flow, G can be expressed as a sum of a gravitational potential and a potential due to capillary suction:

$$G = H(u) + x$$

where $H(u)$ is called a *hydrostatic potential*.

The Model

Combining all of this we obtain the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial x}K(u) \tag{3}$$

Empirical expressions are known for $D(u) := K(u)\dfrac{dH}{du}$, and in particular, we have $D(u) = D_0 u^{m-1}$ and $K(u) = K_0 u^n$, for D_0, K_0, m, n positive constants.

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$$\frac{\partial u}{\partial t} = \frac{\partial u^n}{\partial x} + \frac{\partial^2 u^m}{\partial x^2}, \quad n, m \geq 1,\ x \in \mathbb{R},\ t > 0 \tag{4}$$

Comments on the Model Derivation

- Darcy's Law can be derived from basic principles for porous media, but the derivation is quite involved. If interested, see S. Whitaker, *Flow in Porous Media I: A theoretical derivation of Darcy's Law*. Transport in Porous Media, 1 (1986), 3-25.

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- The continuity equation is derived by considering a unit volume of the medium and equating the rate of increase of the fluid mass within that volume to the net mass flux into the volume.

Movie

Let's watch a simulation of this type of flow!

Click me!

Your task

Your task is to analyze the model (4) by developing analytical solutions.

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You will consider the case when $n = 3, m = 2$. A typical approach to solving such problems is by seeking *traveling wave solutions*; that is, seeking solutions of the form

$$u(x, t) = u(x - ct)$$

which represents a traveling wave with speed c.