

## STUDENT VERSION

### A Gear Train

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#### STATEMENT

In this scenario you will model a series of interfacing gears, often called a gear train (see Figure 3). Your model will involve a second order linear differential equation with constant coefficients. The coefficients will encode the physical properties of the system such as moments of inertia of the gears and rotational friction.

#### **First approximation: gears with no teeth**

We are at this point only interested in the rough geometry of the problem, so let's smooth out the scenario and remove a complication: teeth. To this end, consider two disks of different radii,  $r_1$  and  $r_2$ . Let us suppose that the smaller disk of radius  $r_1$  is driving the larger disk with radius  $r_2$ . If you are able to build a model, fix your disks to a base so that they contact at a point on their circumferences and are able to rotate about their centers. You will rotate the smaller (driving) disk and observe a rotation in the larger (driven) disk. Let  $\theta'_1$  and  $\theta'_2$  be the rates, i.e. angular velocities (revolutions per minute), at which the disks revolve, respectively.

If you have access to a physical model, experiment with it. Otherwise, you may simply imagine such a set up or use an online tool like Gear Generator [3] to aid your investigation. By means of an introduction, let us begin with several questions.

1. In Figure 1, label  $\theta'_1$  and  $\theta'_2$ . Be sure to indicate the direction of motion. For now ignore friction, moments of inertia, or the possibility that the disks will slip.

- Rotate the driving gear clock-wise. The driven gear rotates \_\_\_\_\_ clock-wise.
- As  $\theta'_1$  increases,  $\theta'_2$  will \_\_\_\_\_.
- Make a hypothesis about how a change in the radii would affect the relationship between  $\theta'_1$  and  $\theta'_2$ .

Hypothesis:

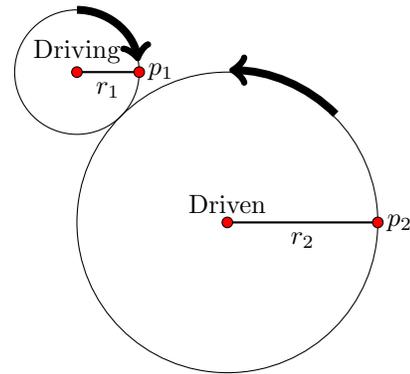


Figure 1. A pair of disks in contact.

2. Consider the points  $p_1$  and  $p_2$  on the circumference of each disk.
- Point  $p_1$  on the circumference of the driving disk moves with linear velocity \_\_\_\_\_.
  - Point  $p_2$  on the circumference of the driven disk moves with linear velocity \_\_\_\_\_.
3. Without loss of generality, we may suppose that points  $p_1$  and  $p_2$  coincide at the point at which the disks contact. Observe that because the disks are in contact, the linear velocities at the point in contact are equal. Alternatively, lengths of equal arc pass through the point of contact. We conclude that

$$\theta'_1 r_1 = \theta'_2 r_2 \text{ or } \frac{\theta'_1}{\theta'_2} = \frac{r_2}{r_1}.$$

- Test your hypothesis from Question 1 above against these facts.
    - When  $r_1 = r_2$ , the angular velocities are \_\_\_\_\_.
    - When  $r_1$  increases, say to  $r_1 = 2r_2$ , the angular velocity  $\theta_1$  is \_\_\_\_\_ of  $\theta_2$ .
4. Observe that  $\theta'_1$  and  $\theta'_2$  increase in opposite directions, one clock-wise and the other counter clock-wise. This is okay, as we have not yet declared the global direction in which the angle increases. In studying the system of disks or gears we will declare that positive angle is measured in the counter clock-wise direction, and therefore,

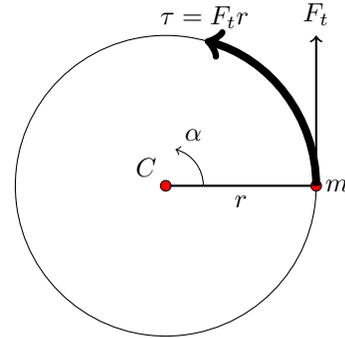
$$\theta'_1 r_1 = -\theta'_2 r_2.$$

You will need to use this information to ensure the model references the correct orientation.

### A review of torque

Suppose a force is applied to a point on the circumference of a disk of mass  $m$ , causing the disk to rotate about its center  $C$ . The product of the tangential component of the force,  $F_t$ , and the radius,  $r$ , is called the *torque*,  $\tau$ . In this rotational setting,  $F_t = ma_t$ , where  $m$  is the mass of the disk and  $a_t$  is the tangential component of acceleration. Since  $a_t$  is the product of the radius  $r$  and the angular acceleration  $\alpha$ , we have

$$\tau = F_t r = (ma_t)r = mr(r\alpha) = mr^2\alpha = I\alpha.$$



**Figure 2.** A rotating disk.

Torque  $\tau$  is proportional to angular acceleration  $\alpha$  and this constant is called the *moment of inertia*  $I$ . It is a measure of the tendency of the disk to resist rotation. In our model, the measure of the angle will be denoted by  $\theta$ , so  $\alpha = \theta''$  and torque will be expressed as  $\tau = I\theta''$ . Thus, accounting for the moment of inertia will lead to a second order differential equation.

### Gears and gear ratios

We will now repeat the setup as introduced above, but with spur gears. You may work with the online tool Gear Generator [3] or with a physical model. Again, we are ignoring moments of inertia, mass, friction, but the gears in the physical model should mesh easily and not slip.

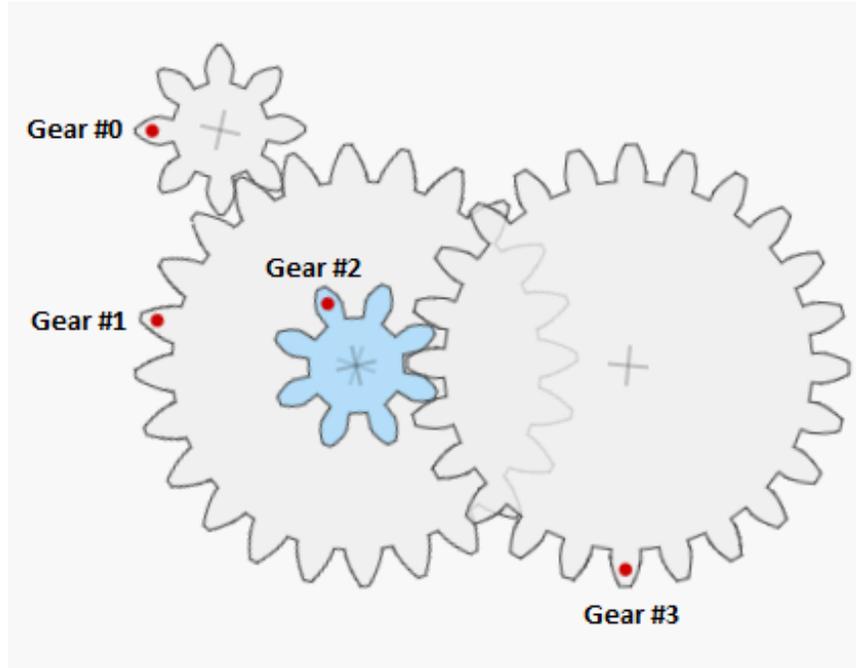
The quotient of the number of teeth on the driven gear to the number of teeth on the driver gear is called the gear or velocity ratio. Let  $n_1 = 8$  and  $n_2 = 24$  be the respective numbers of teeth of each gear. Then,

$$\frac{\# \text{driven}}{\# \text{driver}} = \frac{n_2}{n_1} = \frac{24}{8} = \frac{3}{1} \text{ or } 3 : 1.$$

Thus, every three revolutions of the smaller driving gear will result in one revolution of the larger driven gear.

1. Small gears driving larger gears result in smaller velocities. For instance, if  $n_1 = 45$  rpm, then  $n_2 = 45 \div 3 = 15$  rpm. Conversely, larger gears driving smaller gears increase velocities. With the same gear ratios, but the roles of  $n_1$  and  $n_2$  as driving and driven gears interchanged, we have  $n_2 = 45$  rpm and then  $n_1 = 45 \times 3 = 135$  rpm.
2. In principle, a gear train can include any number of gears. The gears intermediate to the driving gear, connected to a crank or motor, and the final gear of the train are called *idler gears*. We need to multiply the gear ratios of each pair of gears.

We will use Gear Generator [3] to simulate the gear train in Figure 3. Adjust the “Gear properties” to match the number of teeth for each gear as given in the table below. The box marked “Internal Gear” should not be checked. After setting the number of teeth for each of



**Figure 3.** A gear train modeled in Gear Generator. Gear #0 on the left has 8 teeth and rotates clock-wise. Gears #1 and #2, joined at the axis passing through their centers, are idler gears.

the gears, set gear #1 to have a “pitch diameter” of 2 and a “diametral pitch” of 12. The application will then produce the gear train in Figure 1. The 3D-printed model in Figure 2 has the same configuration. Using Gear Generator, set the driving gear #0 to 45 rpm and fill in the information missing in the table.

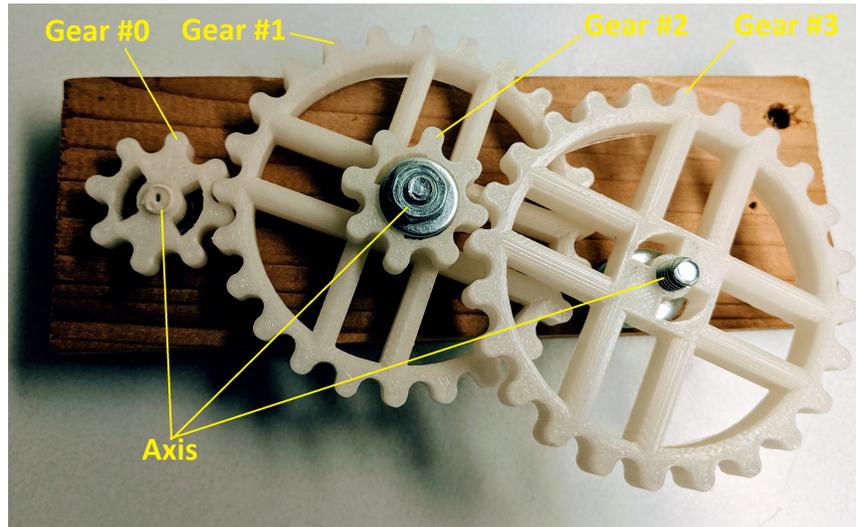
	Gear #0	Gear #1	Gear #2	Gear #3
Teeth	8	24	8	24
Ratio	1:1			
RPM	45			

Observe that the product of the gear ratios of the idler gears #1 and #2 is the gear ratio of gear #3; i.e. in the final analysis of the gear train gear #3 is driven by gear #0.

### Introducing torques and moments of inertia

We will now model a part of the gear train in Figure 3 with a differential equation.

- Let all angles be positive in the counter clock-wise direction.
- How do the signs of  $\theta_0$ ,  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ , the directions in which individual gears of the train rotate, relate to one another?



**Figure 4.** A 3D printed gear train model. Gear #0 on the left has 8 teeth and rotates clock-wise, driven by a motor. Gears #1 and #2, joined at the axis passing through their centers, are idler gears.

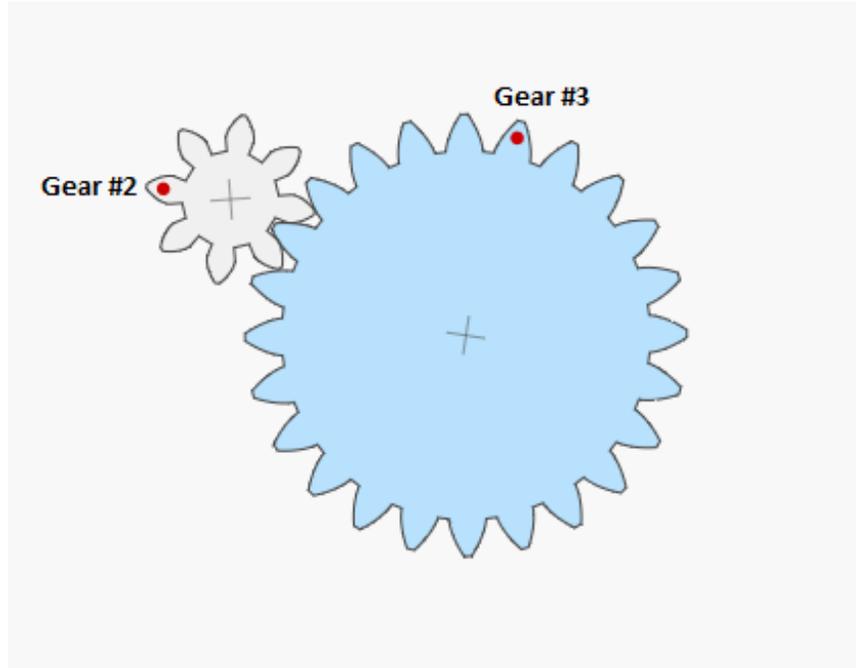
- We will remove gear #0 from the model. As it is driven by a motor, it will serve as the torque input  $\tau$  in the model. Since gears #1 and #2 are fused together, we may proceed with just the pair #2 and #3. See Figure 5.
- What additional parameters should be considered?

### Balancing the torques

Use the information below to label each gear in Figure 5 with forces and torques.

- Label gear #2 with:
  - Torque  $\tau$ , due to the motor driving gear #0, is applied to gear #2 in the direction of its rotation  $\theta_2$ .
  - Tangential to the direction of rotation, a contact force  $F_c$  at the point of contact with gear #3 results in the torque  $F_c r_2$ .
  - Opposite the direction of rotation, the torque  $I_2 \theta_2'$ .
- Label gear #3 with:
  - Tangential to the direction of rotation, a contact force  $F_c$ , equal and opposite to the contact force on gear #2 at the point of contact with gear #2, results in the torque  $F_c r_3$ .
  - Opposite the direction of rotation, the torque  $I_3 \theta_3'$ .
  - Opposite the direction of rotation,  $\beta_r \theta_3'$ , where  $\beta_r$  represents rotational friction.

According to Figure 5:



**Figure 5.** A pair of gears. Gear #2 on the left is driving gear #3

- Acting on gear #2, torques \_\_\_\_\_ and \_\_\_\_\_ face in the same direction.
- Acting on gear #3, torques \_\_\_\_\_ and \_\_\_\_\_ face in the same direction.

D'Alembert's principle is an alternative form of Newton's second law of motion, i.e.  $F = ma$ . It states that a body is in equilibrium under the action of a real force  $F$  and a fictitious force  $-ma$ , or equivalently that  $F - ma = 0$ . You have already turned the forces acting on the gears in Figure 5 into torques. Then, by D'Alembert's principle the torques acting on each of the gears must sum to zero. Write a differential equation describing the rotation of each gear.

- Gear #2:
  
- Gear #3:

From the orientation we have imposed on the problem, we have  $\theta_2 r_2 = -\theta_3 r_3$ . Use this information to eliminate  $F_c$  and  $\theta_2$  in your equations and obtain

$$\left( I_2 + I_3 \left( \frac{r_2}{r_3} \right)^2 \right) \theta_3'' + \beta_r \left( \frac{r_2}{r_3} \right)^2 \theta_3' = \tau. \quad (1)$$

**Solving the differential equation**

- Before delving into the particulars of our model, we can develop a general feeling for a solution of an equation of this type. Solve  $Ay'' + By' = C$  and describe the behavior of the solution.
- Equation (1) is of first order in the variable  $\theta'_3$ . Solve it for  $\theta'_3$  using first order methods.
- Obtain  $\theta_3$ , in terms of the initial conditions  $\theta_3(0)$  and  $\theta'_3(0)$ . [Hint: Remove the arbitrary constants introduced by integration.] Then solve the associated initial value problem by choosing reasonable initial values of  $\theta_3$  and  $\theta'_3$ .
- With the aid of a computer, run some example calculations to describe how the parameters  $I_2$ ,  $I_3$ , and  $\beta_r$  affect the solution.

**REFERENCES**

- [1] Cheever, Erik. Gears and Systems with both Rotation and Translation. <https://lpsa.swarthmore.edu/Systems/MechRotating/RotMechSysGears.html> and <https://lpsa.ttu.swarthmore.edu/Systems/MechRotating/RotMechSysGears.html>. Accessed on 19 August 2019.
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