STUDENT VERSION

Tiling an $n \times 3$ Hallway with 1×2 Tiles

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GOAL

In this exercise, you will explore difference equations in the context of tiling hallways. You will calculate solutions to your difference equation by iteration and become familiar with a helpful tool known as the shift operator.

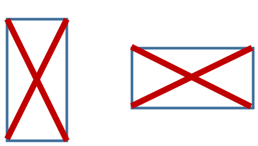
STATEMENT

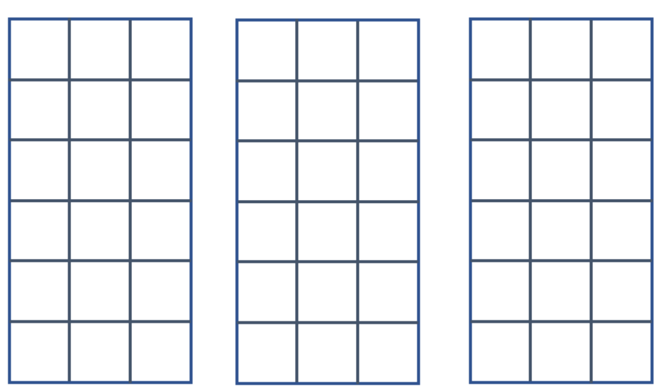
Work has been slow for Kyle the Tile Guy lately. With all the new wood and laminate flooring options, Kyle is not in demand. To pass the time, Kyle, a double major in math and construction engineering was reading one of his old math textbooks and ran across the following problem:

> How many ways can the floor of a hallway that is n units long and 3 units wide be tiled with tiles, each of which is 2 units by 1 unit? Tiles can be placed vertically or horizontally. Kyle was intrigued…how about you? [1, p. 89]

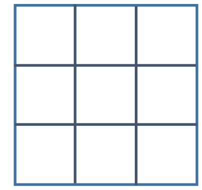
Part 1

1) Draw a few different tiling pictures of $n \times 3$ hallways for $n = 6$.
   Hint: Draw a big "X" to represent the two by one tiles.

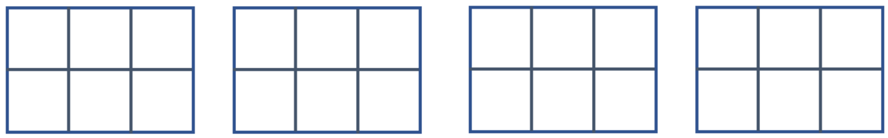




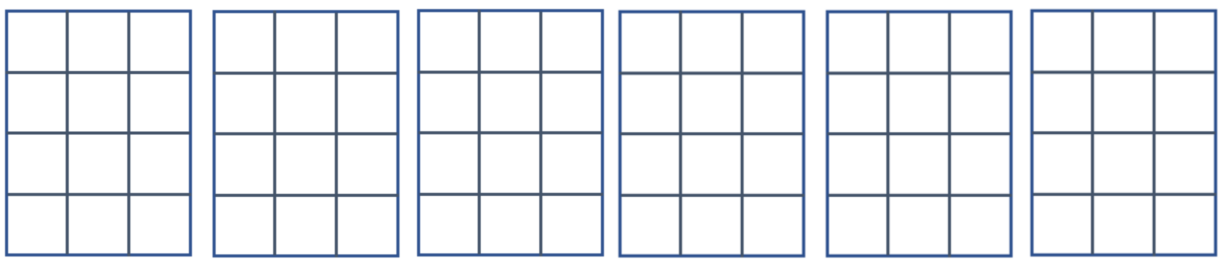
2) Draw a tiling picture of $n \times 3$ hallways for $n = 3$ in the figure to the right. What happened?



3) How many different patterns are possible for a 2×3 hallway?

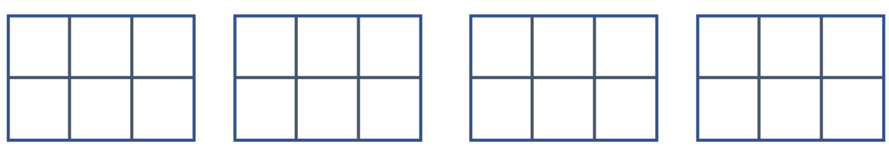


4) How many different patterns are possible for a 4×3 hallway? Hint: Think about symmetry and the patterns developed in the $n = 2$ case so you don't have to draw each one.



5) Define $y(i), i \geq 2$, (for even i) to be the number of ways that an $i \times 3$ hallway can be tiled with the 1×2 tiles. Find: $y(2) =$ ________ and $y(4) =$ _________.

6) If we had an 2×3 hallway, how many ways could the first row be covered by the tiles?



Part 2

7) Define the function z such that $z(n + 2)$ is the number of ways to tile the middle hallway in Figure 1. How many ways could the third hallway be tiled? Hint: Read carefully and remember symmetry!

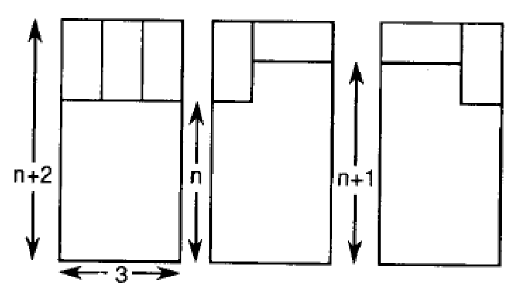
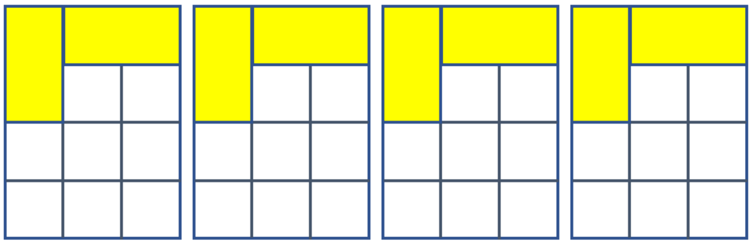


Figure 1: Initial Tiling Options

8) Recall that $y(n+2)$ would be the number of ways that an $(n+2) \times 3$ hallway can be tiled with the 1×2 tiles. Write an equation using the three pictures in Figure 1 for $y(n+2)$.

9) How many patterns are possible for the next step of the middle hallway in Figure 1? Draw the pictures.



10) Write an equation for $z(n+2)$ using the pictures you drew in (9) above.

Part 3

We wish to calculate several values of $y(n)$ by iteration using the equations for $y(n+2)$ and $z(n+2)$.

11) Combine the two equations in (8) and (10) into one by eliminating the $z(n)$ terms.

12) Given we know $y(2) = 3$ and $y(4) = 11$, find $y(6), y(8),$ and $y(10)$.

Part 4

We wish to calculate a closed form expression of $y(n)$ using the equations for $y(n+2)$ and $z(n+2)$ and something called the **shift operator**. [1, p. 17]

13) Define the **shift operator *E*** such that $Ey(n) = y(n+1)$. Similarly, $y(n+2) = Ey(n+1) = E(Ey(n)) = E^2y(n)$ where we define E^2 to be E composed with E. Use the shift operator to reduce the two equations in (8) and (10) into expressions of $y(n)$ and $z(n)$.

14) Use the elimination method of linear equations to eliminate $z(n)$. What is the result?

15) This "characteristic" equation can be solved just like differential equations, but the solutions are in the form E^n instead of e^{rt}.

The solutions of $E^4 - 4E^2 + 1 = 0$ are $E = \pm\left(2+\sqrt{3}\right)^{\frac{1}{2}}$ and $E = \pm\left(2-\sqrt{3}\right)^{\frac{1}{2}}$. Recalling n is even, we obtain the general solution:

$$y(n) = A\left(2+\sqrt{3}\right)^{\frac{n}{2}} + B\left(2-\sqrt{3}\right)^{\frac{n}{2}}.$$

where A and B are some constants. Using the initial conditions $y(2) = 3$ and $y(4) = 11$, find the particular solution:

16) Find $y(6), y(8),$ and $y(10)$ and compare with solutions in (12).

REFERENCES

[1] Peterson, A., and W. Kelley. 1991. *Difference Equations: An Introduction with Applications.* San Diego: Academic Press.