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STUDENT VERSION
Heat Diffusion

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STATEMENT

Background

The one-dimensional heat (diffusion) equation,

$$u_t = ku_{xx}, \qquad x \in (0, L), \qquad t > 0, \tag{1}$$

is a standard topic considered in introductory Partial Differential Equation courses. In (1), $u = u(x, t)$ is a scalar function that represents the temperature (°C) at any time t (s) and position x (mm) measured from the left end of a long thin rod of length L. The parameter k represents the thermal diffusivity of the rod material; for steel, $k = 4$ mm^2/s, while aluminum has a thermal diffusivity of $k = 97$ mm^2/s, corresponding to its propensity to conduct heat. The possible boundary conditions at $x = 0$ and $x = L$ include prescribed temperatures (Dirichlet), insulation (Neumann), and Newton's Law of Cooling (Robin). Besides being an accessible entry point for modeling a physical phenomenon with differential equations and introducing separation of variables as a fundamental analytical technique for solving PDEs, (1) provides the basic structure for studying the larger class of parabolic equations.

Credit for the formulation of (1) and its solution belong to Jean-Baptiste Joseph Fourier. At the beginning of the nineteenth century, Fourier proposed analytical solutions which were sums of sine and cosine functions, what we now call Fourier series [?, ?]. At the time, the veracity of Fourier's heat diffusion model and solution technique were heavily doubted but have since been bolstered by further analysis [?]. History has made him a household name, at least for some mathematicians, because of the pervasiveness and power of his work. Other applications that can be modeled by

(1) include the spreading of a pollutant in water [?], financial option pricing [?] and digital image restoration [?]. Solution methods based on Fourier series are powerful tools in analyzing particular PDEs on bounded domains.

Experiment

The physical phenomenon of heat diffusion in a long thin metal rod is measurable with the experimental design illustrated in Figure 1. The ends of the rod are submerged in water baths at different temperatures and the heat from the hot water bath travels through the metal to the cooler end. The temperature of the rod is measured at four locations; those measurements are sent to a Raspberry Pi, which processes the raw data and sends the collated data to be displayed on the computer screen.

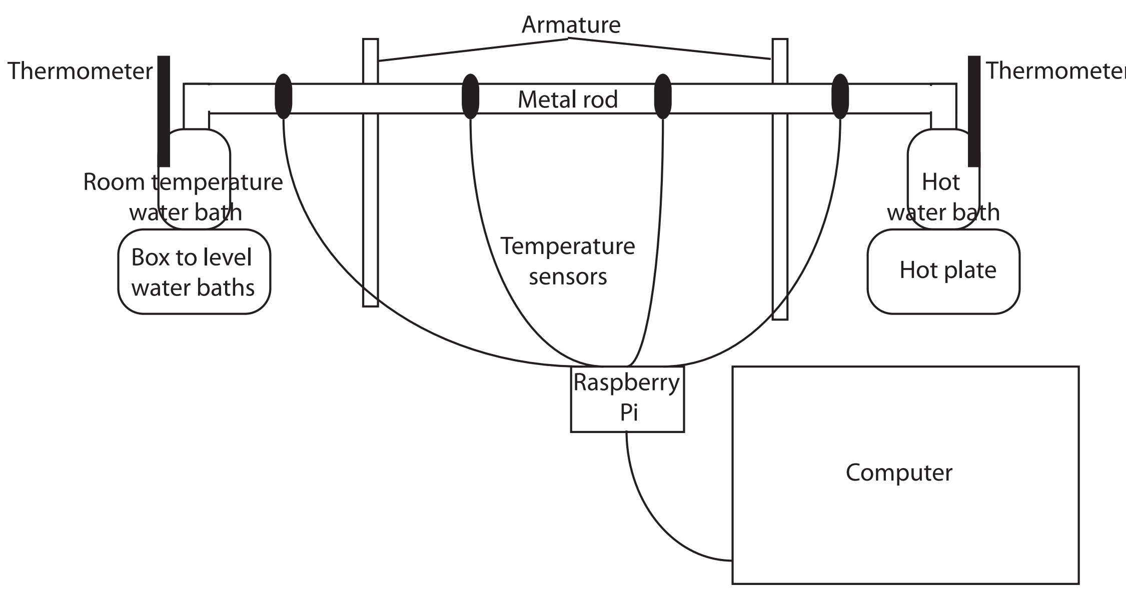

Figure 1: The experimental apparatus to observe heat diffusion

If you are collecting your own data, make initial measurements of the length and width of the rod, as well as where the temperature sensors are. Sketch your experimental setup so that you can refer to it later. Before collecting any data, consider the following questions:

- What type of metal is your rod made of? What is its expected thermal diffusivity?
- How well do you think the metal rod in front of you corresponds to one spatial dimension?
- What are the boundary conditions in this set-up? That is, what are the temperatures of the water baths at each end of the rod? Do you think they will remain fixed throughout the demonstration or do you think they will fluctuate? Why?
- For short and long periods of time, what do you expect to happen when you submerge the ends of the rod in each water bath?

Use the code provided in `PythonCodeRecord.pdf` to start recording data from the temperature sensors. Then submerge the ends of the rod in the water baths. The code provided in `PythonCodeGraph.pdf` creates a graph of temperature against time for each of the temperature sensors. Collect data until the graph exhibits plateaus at each sensor location for a few hundred seconds. Stop the data recording and save the collected temperatures to use in answering the questions below.

If you are not collecting your own data, see the provided Excel file `HeatEquationData.xls` in which you can find the time and temperature data that are shown in Figure 2. To produce this data set, we used an aluminum rod of length $L = 300$ mm and square cross-sectional width 3.2 mm. Foam tubing, with a thickness of 25 mm, was wrapped around the rod and sensors to provide some insulation. The ambient temperature in the room was 24°C and the hot water bath measured 53°C.

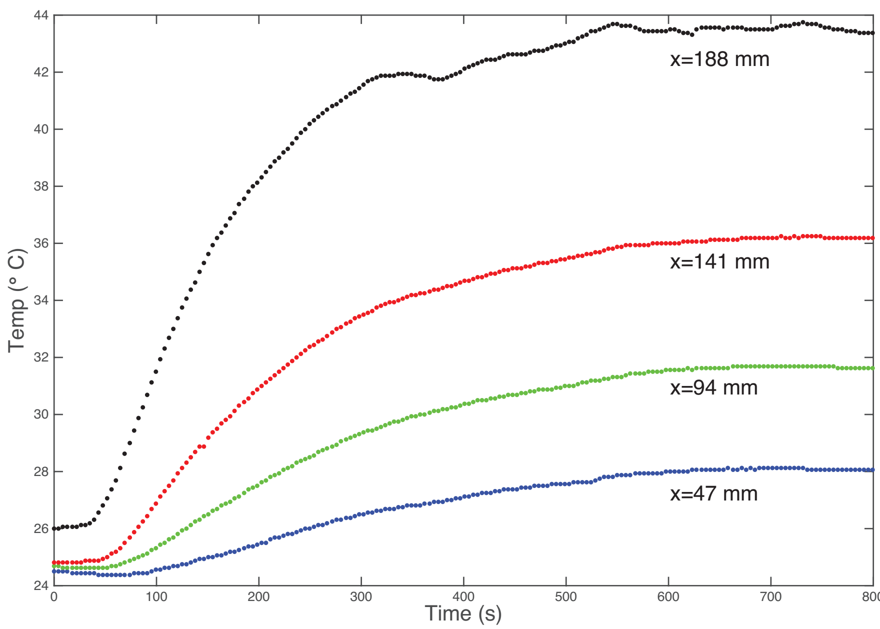

Figure 2: Temperature readings from four sensors along the rod over the course of the heat diffusion demonstration [?]

Mathematical Model

The following assumptions are made in the derivation of (1):

- the rod is perfectly insulated so no heat is lost through the surface;
- the rod material is uniform;

- there is no internal heat generation;
- there is no heat applied to the rod on the lateral surface, only on the ends.

Reflect on whether these four assumptions are met, approximated, or neglected in the experiment carried out above. How do you think the degree to which these assumptions match the experimental setup will affect the model's accuracy? See [?] for more details regarding assumptions and governing equations for the model.

Analysis

1. Write the initial boundary value problem for the heat equation corresponding to the experimental setup observed.
2. Determine the steady-state solution, $u_s(x)$, of the initial boundary value problem you wrote above.
3. Explain how and why the analytical technique of separation of variables, i.e. assuming $u(x,t) = X(x)T(t)$, fails with the initial boundary value problem above.
4. Instead of solving for $u(x,t)$, consider the function $v(x,t) = u(x,t) - u_s(x)$, which measures the displacement of the time-dependent solution from the steady-state solution. Write the associated initial boundary value problem for $v(x,t)$.
5. Use separation of variables to solve for $v(x,t)$. Explain why the technique is successful with the initial boundary value problem for v, in contrast to the initial boundary value problem for u.
6. Now determine the solution $u(x,t)$ of the first initial boundary value problem.

Numerics

1. Use Matlab to graph the Fourier series solution for $u(x,t)$ that you obtained in Step (6) of the previous section on Analysis. Discuss the choices necessary to do this.
2. Read Matlab's help documentation and then use its PDE solver `pdepe` to simulate the solution of the full one-dimensional heat equation problem corresponding to the experimental setup above. Describe how this numerical simulation compares with the plot of your analytical solution, particularly near the initial condition. Discuss a possible cause for the difference between the two plots.
3. Overlay the data points from the four temperature sensors in the experiment on top of the simulations from the previous question. Discuss how well (or not) the experimental data points match up with the numerical solution. Compare the data and model over time.

Errors and Fine-Tuning

1. Describe some possible sources of error between the physical experiment and mathematical model.

2. Identify any parameters that could be changed to fine-tune the model so that the model better matches the data points. Explain your choices.
3. In Matlab, experiment with values for the parameters you identified above to see if there is an improvement in how well the model fits the experimental data. Summarize your findings.