Scudem 2019, Problem A

November 2019

1 Introduction

We are working on problem A.

1.1 Assumptions

- People can influence the styles of others
- The amount a person is influenced by someone else is a function of the difference between the styles of the two people.
- A person can 'pull' another person's style to be more similar to their own style or 'push' the person's style farther away.
- We can measure a person's fashion style using a vector that represents the person's location along some number of 'axes of fashion'.
- People can be either a conformist, a non-conformist, or an average person, and this completely describes their behavior.
  - Conformists will imitate the largest group of people that dress similar to themselves.
  - Non-conformists will attempt to dress differently from anyone by changing their style to be different than those with the most similar style.
  - Average people will imitate people with similar styles, but will change their own style if they are too similar to those around them.

1.2 Model

Let $V = \{\mathbf{v}_1(n), \mathbf{v}_2(n), \ldots, \mathbf{v}_k(n)\}$ be a collection of n-dimensional vectors that represent the sartorial styles of the k different people we are observing, where a person's style is a function of discrete time n.

Example 1.1. Assume that we are examining 3 people and that they dress based on 3 axes of fashion–shirt, pants, and shoes. Then, for example, we may represent their styles as $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$, where each position in the vector represents shirt, pants, and shoes respectively. Then person 1 and person 3 wear the same style of shoes, and person 2 and person 3 wear the same style of pants. Since $||\mathbf{v}_1 - \mathbf{v}_2|| = \sqrt{6}$, $||\mathbf{v}_1 - \mathbf{v}_3|| = \sqrt{20}$, and $||\mathbf{v}_2 - \mathbf{v}_3|| = \sqrt{26}$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ have styles that are overall much more similar to each other than they are to $\mathbf{v}_3$ (where $||\mathbf{v}||$ is the Euclidean distance of $\mathbf{v}$).

We first assume that a person can react by moving their own style closer or further away from the other person along the linear path defined by their own style and the other person's style. We then say that g_i is a function of the distance between 2 people's style that describes the direction person i moves (closer or further) and how far they move (specifically, it gives what fraction of the distance between the 2 people that person i moves). For simplicity, we assume $g_i(x)$ is continuous for $x \geq 0$, but it does not need to be differentiable. Lastly, we assume that the distance they move is equal to the output of g_i. This gives us the following system of k different discrete dynamical equations

$$\mathbf{v}_i(n+1) = \mathbf{v}_i(n) + \sum_{\mathbf{w}(n)\in V\setminus\{\mathbf{v}_i(n)\}} g_i\left(||\mathbf{w}(n) - \mathbf{v}_i(n)||\right)\left(\mathbf{w}(n) - \mathbf{v}_i(n)\right) \tag{1.1}$$

We then transform this into a system of continuous differential equation,

$$\begin{aligned}
\mathbf{v}_i(n+1) =&\ \mathbf{v}_i(n) + \sum_{\mathbf{w}(n)\in V\setminus\{\mathbf{v}_i(n)\}} g_i\left(||\mathbf{w}(n) - \mathbf{v}_i(n)||\right)\left(\mathbf{w}(n) - \mathbf{v}_i(n)\right)\\
\mathbf{v}_i(n+1) - \mathbf{v}_i(n) =& \sum_{\mathbf{w}(n)\in V\setminus\{\mathbf{v}_i(n)\}} g_i\left(||\mathbf{w}(n) - \mathbf{v}_i(n)||\right)\left(\mathbf{w}(n) - \mathbf{v}_i(n)\right)\\
\mathbf{v}_i(t+\Delta t) - \mathbf{v}_i(t) =& \left(\sum_{\mathbf{w}(t)\in V\setminus\{\mathbf{v}_i(t)\}} g_i\left(||\mathbf{w}(t) - \mathbf{v}_i(t)||\right)\left(\mathbf{w}(n) - \mathbf{v}_i(n)\right)\right)\Delta t\\
\lim_{\Delta t\to 0}\frac{\mathbf{v}_i(t+\Delta t) - \mathbf{v}_i(t)}{\Delta t} =& \sum_{\mathbf{w}(t)\in V\setminus\{\mathbf{v}_i(t)\}} g_i\left(||\mathbf{w}(t) - \mathbf{v}_i(t)||\right)\left(\mathbf{w}(n) - \mathbf{v}_i(n)\right)\\
\frac{d(\mathbf{v}_i(t))}{dt} =& \sum_{\mathbf{w}(t)\in V\setminus\{\mathbf{v}_i(t)\}} g_i\left(||\mathbf{w}(t) - \mathbf{v}_i(t)||\right)\left(\mathbf{w}(n) - \mathbf{v}_i(n)\right).
\end{aligned}$$

We can then write out the sum to obtain

$$\begin{aligned}
\frac{d(\mathbf{v}_i(t))}{dt} =&\ g_i\left(||\mathbf{v}_1(t) - \mathbf{v}_i(t)||\right)\mathbf{v}_1(t) + \cdots + g_i\left(||\mathbf{v}_{i-1}(t) - \mathbf{v}_i(t)||\right)\mathbf{v}_{i-1}(t)\\
&- \Big(g_i\left(||\mathbf{v}_1(t) - \mathbf{v}_i(t)||\right) + \cdots + g_i\left(||\mathbf{v}_{i-1}(t) - \mathbf{v}_i(t)||\right)\\
&+ g_i\left(||\mathbf{v}_{i+1}(t) - \mathbf{v}_i(t)||\right) + \cdots + g_i\left(||\mathbf{v}_k(t) - \mathbf{v}_i(t)||\right)\Big)\mathbf{v}_i\\
&+ g_i\left(||\mathbf{v}_{i+1}(t) - \mathbf{v}_i(t)||\right)\mathbf{v}_{i+1}(t) + \dots\\
&+ g_i\left(||\mathbf{v}_k(t) - \mathbf{v}_i(t)||\right)\mathbf{v}_k(t),
\end{aligned} \tag{1.2}$$

and from this we can represent the system as a matrix. We let $g_i\left(||\mathbf{v}_j(t) - \mathbf{v}_i(t)||\right) = g_{i,j}$ for brevity. We then let

$$A = \begin{bmatrix}
-\sum_{m=2}^{k} g_{1,m} & g_{1,2} & g_{1,3} & \cdots & g_{1,k}\\
g_{2,1} & -g_{2,1} - \sum_{m=3}^{k} g_{2,m} & g_{2,3} & \cdots & g_{2,k}\\
g_{3,1} & g_{3,2} & -\sum_{m=1}^{2} g_{3,m} - \sum_{m=4}^{k} g_{3,m} & \cdots & g_{3,k}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
g_{k,1} & g_{k,2} & g_{k,3} & \cdots & -\sum_{m=1}^{k-1} g_{k,m}
\end{bmatrix}$$

and $W = \begin{bmatrix}\mathbf{v}_1(t)\\ \vdots\\ \mathbf{v}_k(t)\end{bmatrix}$. This gives us

$$A \times W = W'. \tag{1.3}$$

We now have the option to simplify the model. We do this by letting $g_1 = g_2 = \cdots = g_k$. Then, since Euclidean distance is a norm on Euclidean n-space,

$$\begin{aligned} ||\mathbf{v}_j(t) - \mathbf{v}_i(t)|| =& ||\mathbf{v}_i(t) - \mathbf{v}_j(t)|| \\ g_i\left(||\mathbf{v}_j(t) - \mathbf{v}_i(t)||\right) =& g_j\left(||\mathbf{v}_i(t) - \mathbf{v}_j(t)||\right) \\ g_{i,j} =& g_{j,i}, \end{aligned}$$

so A is symmetric.

2 Analysis of the Symmetric Case

In this case, we assume $g_1 = g_2 = \cdots = g_k$ which implies that A is symmetric. We let $g_1 = \cdots = g_k = g$.

2.1 Equilibria

Equation 1.3 implies that to find equilibrium solutions, we simply need to find null(A). Clearly, A is not full rank, so the nullspace has at least 1 dimension by the rank-nullity theorem; we can find the nullspace by decomposing A. If we let $B = -A$, then equation 1.3 is equivalent to $B \times -W = W'$. Then since B is a scalar multiple of A, null(B) = null(A). And since B has positive entries along its diagonal, it is positive definite; this means we may perform a Cholesky decomposition on it. We may then write $B = LL^T$, where L is a lower triangular matrix Since null(L^T) = null(LL^T) = null(B) = null(A), we now only need to calculate and find the nullspace of L^T. To calculate L, we use the following formulae:

$$L_{j,j} = \sqrt{B_{j,j} - \sum_{m=1}^{j-1} L_{j,m}^2} \tag{2.1}$$

$$L_{i,j} = \frac{1}{L_{j,j}} \left(B_{i,j} - \sum_{m=1}^{j-1} L_{j,m} L_{i,m} d_m \right), \text{ for } i > j. \tag{2.2}$$

We may then directly calculate the nullspace of L^T.

However, since A is comprised of functions of $\mathbf{v}_1(t), \mathbf{v}_2(t), \ldots, \mathbf{v}_k(t)$, this is not a linear system in general and the nullspace will depend entirely on the values of g. Though finding and quantitatively characterizing all possible equilibria exceeds the scope of this competition, we can still characterize some equilibria and their stability.

2.2 Stability

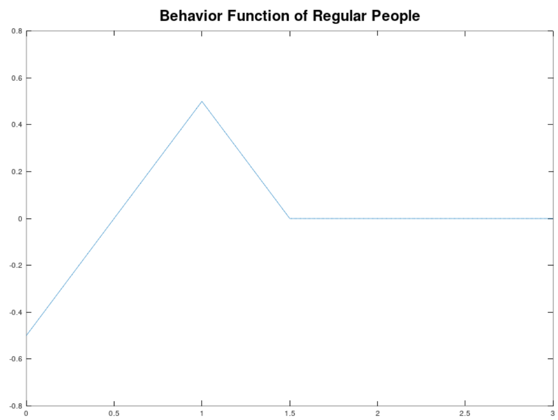
One obvious basis vector for the nullspace is $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, which represents the situation where each person has the same style. The stability of this solution will depend on the behavior of g. Specifically, it is stable if and only if $g(0) > 0$.

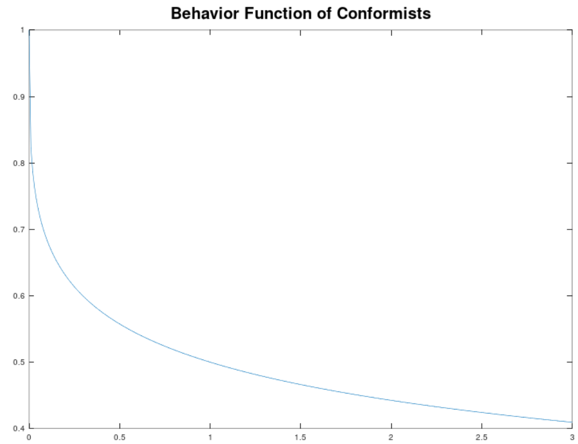
Secondly, if we choose g to be a function that behaves in a certain way, we can find infinitely many equilibria by examining the behavior of g.

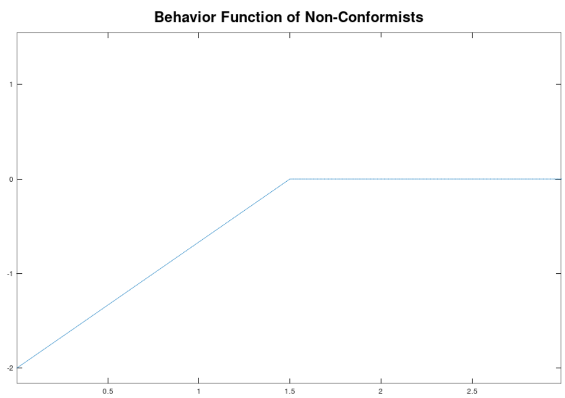
Example 2.1. Let

$$g(x) = \begin{cases} 1x - .5 & 0 < x \leq 1 \\ -1x + 1.5 & 1 < x \leq 1.5 \\ 0 & 1.5 < x, \end{cases}$$

then an equilibrium occurs any time people's style are all more than 1.5 units away from each other. Additionally, a stable equilibrium occurs when every person is exactly .5 away from each other. This kind of equilibrium can only occur when styles have n dimensions and there are $n + 1$ or fewer people, as $n + 1$ is the maximum number of equidistant points in n-space. There are, of course, other stable equilibria given this g, but as before, this exceeds the scope of this project.







3 Analysis of Non-Symmetric Case

The case examined above assumes that each g_i is equal, equivalent to assuming that all people in the model behave the same. This is a very restrictive assumption, so we will now examine the case were there is at least one person that acts differently. This immediately removes the symmetry of A; depending on the differences in behavior, it can also make A full rank most of the time. Because of this, any general analysis of equation 1.3 far exceeds the scope of this summary. We may, however create different $g_1, g_2, \ldots, g_k$ and observe some simulations.

We let $g_1 = g_2 = \cdots = g_i = g$ defined in example 2.1; we let $g_{i+1} = \cdots = g_j = f$, where

$$f(x) = \begin{cases} \frac{4}{3}x - 2 & 0 < x \leq 1.5 \\ 0 & 1.5 < x; \end{cases}$$

and we let $g_j = \cdots = g_k = c$, where

$$c(x) = \frac{1}{\sqrt[3]{x} + 1}.$$

Then g describes the behavior of the average person, f describes the behavior of a nonconformist, and c describes the behavior of a conformist.

We will examine the behavior of each group individually then consider what happens when they are together. Under these circumstances, the non-conformists will push each other away until they are spread out enough. Conformists will approach their nearest large group; since $c(x) > 0$ for any x, a finite amount of conformists will always eventually gravitate towards a single point where all of them are located.

Average people are more intricate. If they begin too close to each other, then two things can happen. They may behave chaotically, as their group begins to convulse until it breaks apart into smaller groups that may recombine by smashing together then breaking apart again, repeating this process until the groups move too far away to interact. Otherwise, these convulsions will abate and approach an equilibrium. If each person starts far enough away from others, nothing will happen. If they start in different enclaves of nearby people, these enclaves will behave independently of each other.

Supposing there are conformists and average people, then the conformists will move towards the nearest large group of conformists or average people. The average people will move away from conformists, so if the space is unbounded, conformists may chase average people without bound and never reach an equilibrium. Otherwise, if one or more conformists are located at the center of an enclave of average people, then this may approach a stability point. Conformists can either start inside of an enclave or, if they have enough speed, they may pierce through to the center of an enclave before it can move away.

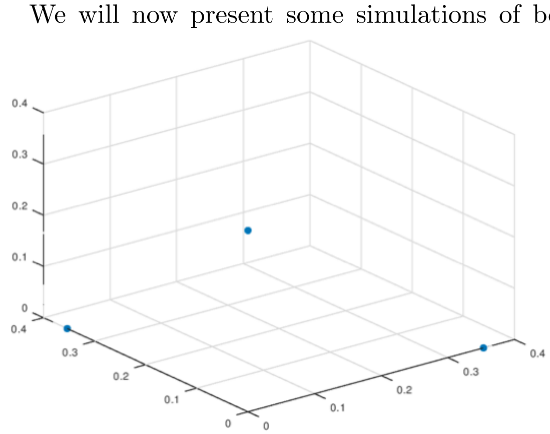
If there are conformists and non-conformists, conformists may again give chase, possibly preventing an equillibrium. Any equilibrium that does arise will have all conformists at the center, surrounded by the non-conformists.

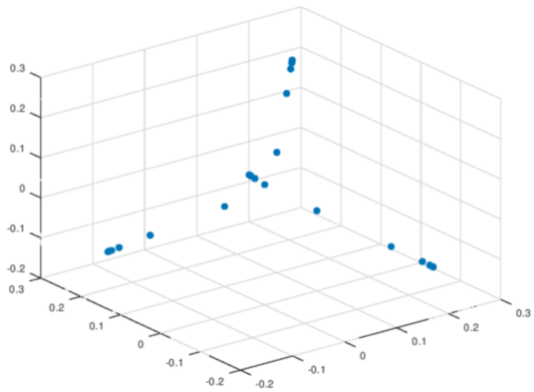
Conformists and average people together will behave largely the same as when they are alone. Non-conformists will spread out until they no longer affect the average people, at which point the average people will behave as if the non-conformists are not present.

If all three are present, then any equilibria will again look like conformists at the very center, surrounded by all of the average people, and then non-conformists at the outskirts.

4 Simulations

We will now present some simulations of both the symmetric case and the non-symmetric case.

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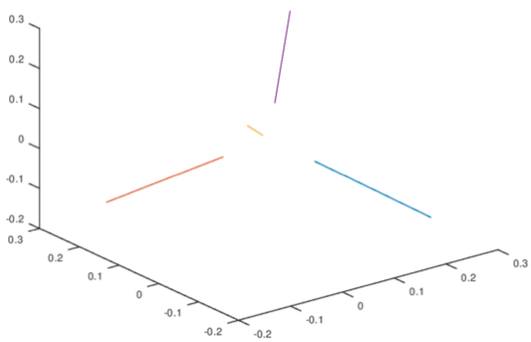
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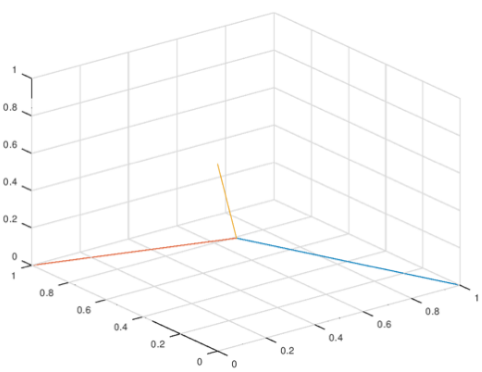
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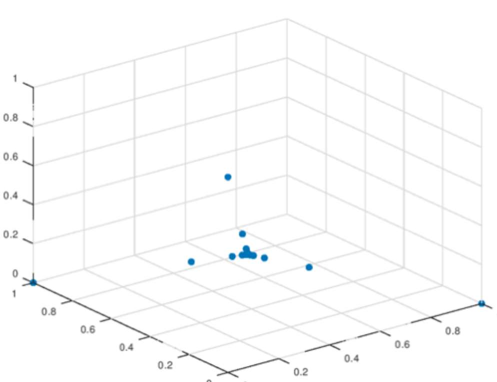
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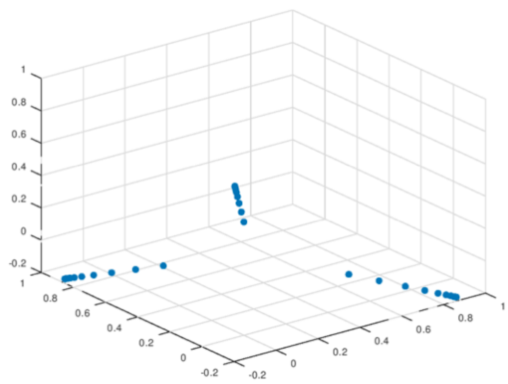
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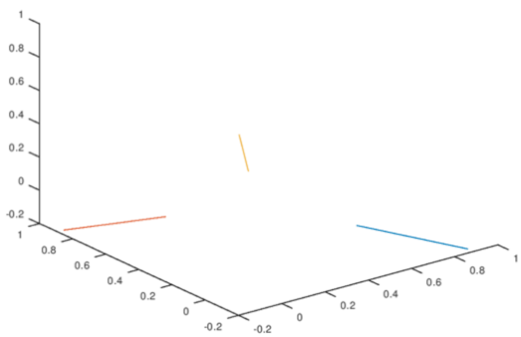
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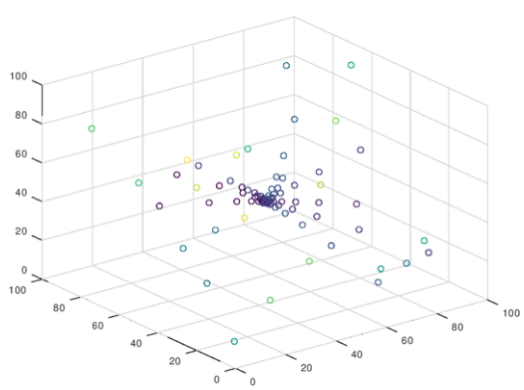
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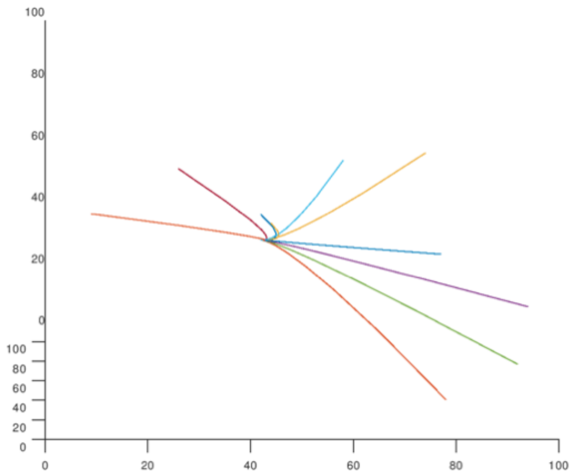

5 Future Considerations

Additional assumptions that would make the model stochastic to incorporate later:

- A person's style can mutate unpredictably.
- Behavior is on a continuum from nonconformists to average people to conformists.
- A person can move along the continuum and will do so sporadically.