Problem C: Chemical Espionage

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Some insects, including the large white cabbage butterfly *Pieris brassicae*, emit pheromones to attract potential male partners. While this is necessary, sometimes too many male butterflies respond to this chemical signal making it difficult for the female to focus on finding a safe place to lay her eggs. In response, males excrete anti-aphrodisiacs to dissuade other male butterflies from approaching the female. However, a species of parasitic wasp has picked up on these chemical signals so that they follow the female butterfly. When she lays her eggs, the wasps will lay their own eggs in the butterflies' eggs. Consequently, there are tradeoffs for the butterflies in using these anti-aphrodisiacs. We attempt to model these tradeoffs, whether the two species can coexist or if one pushes the other out.

Before we begin building our model, we must first make some assumptions about the two species. Our first assumption is that exactly half of the butterfly population is female while the other half is male. The second assumption is that without wasps the butterfly population grows without bound, but without butterflies the wasp population will not grow at all. That is the only way for the wasp population to grow is by laying their own eggs in the butterflies'. Our last assumption is that butterflies go straight from eggs to butterflies; there are no intermediate stages. Finally, our model is in the form of days.

In building our model, we decided to have two equations: the butterfly population and the wasp population. The equation we came up with to model the change in butterfly population is $\frac{dB}{dt} = aB\left(1 - \frac{W}{k}\right)$ where B represents the butterfly population. Here a is the butterfly growth rate multiplied by the butterfly population. Then we added a second term to describe how the number of wasps in the system will affect this growth rate (some wasps will take away from the growth of the butterfly population by laying their own eggs in the butterflies'). To model this, we figured there must be some critical number k that represents the number of wasps where every butterfly egg laid is instead injected with a wasp egg. To make this affect the growth rate in a reasonable fashion we thought of it logistically, $(1 - \frac{W}{k})$, where W is the number of wasps. When $W < k$ this term is close to one, so the change in number of butterflies is close to the natural growth of aB. Conversely, when $W > k$, the term becomes negative, and therefore the growth of butterflies becomes negative. Lastly, when $W = k$, the change in butterfly population is 0.

The equation we came up with for the change in wasp population is $\frac{dW}{dt} = \frac{1}{2}cBW - mW$. We started by multiplying $\frac{1}{2}cBW$. The growth of the wasp population depends on the number of wasps and the number of female butterflies that they can follow to the eggs. This gave us $\frac{1}{2}BW$, remembering that we assumed exactly $\frac{1}{2}$ of the butterfly population is female. We also considered that a butterfly lays more than one egg and added a parameter c to account for this and the probability that a butterfly egg will be injected by a wasp. The second term in $\frac{dW}{dt}$ is $-mW$ where m is the death rate of wasps.

Our goal is to know how these two populations interact in the long run. To find this long-term behavior, we need to know the equilibrium points, where $\frac{dB}{dt}$ and $\frac{dW}{dt}$ both equal zero. Solving for this gave us two points: $(0,0)$ and $(\frac{2m}{c}, k)$ representing extinction and coexistence respectively.

To know how these two solutions play out in the long run, we analyze the stability of each by finding the eigenvalues for the Jacobian matrix of the system. Starting with our extinction equilibrium $(0,0)$, we find the eigenvalues to be a and $-m$. The solution is stable if all eigenvalues are negative. The second eigenvalue, $-m$, is negative. However, the eigenvalue a is not because we designed our model so all parameters would be positive. This makes the extinction equilibrium an unstable solution.

Examining our coexistence equilibrium $(\frac{2m}{c}, k)$ we find the eigenvalues to be $\pm\sqrt{am}$. Again, one of these eigenvalues is negative. That makes the coexistence equilibrium unstable as well. This leads us to the phenomenon known as the principle of competitive exclusion. To determine the long-term behavior of this system with inconclusive solutions, we must instead perform a graphical analysis.

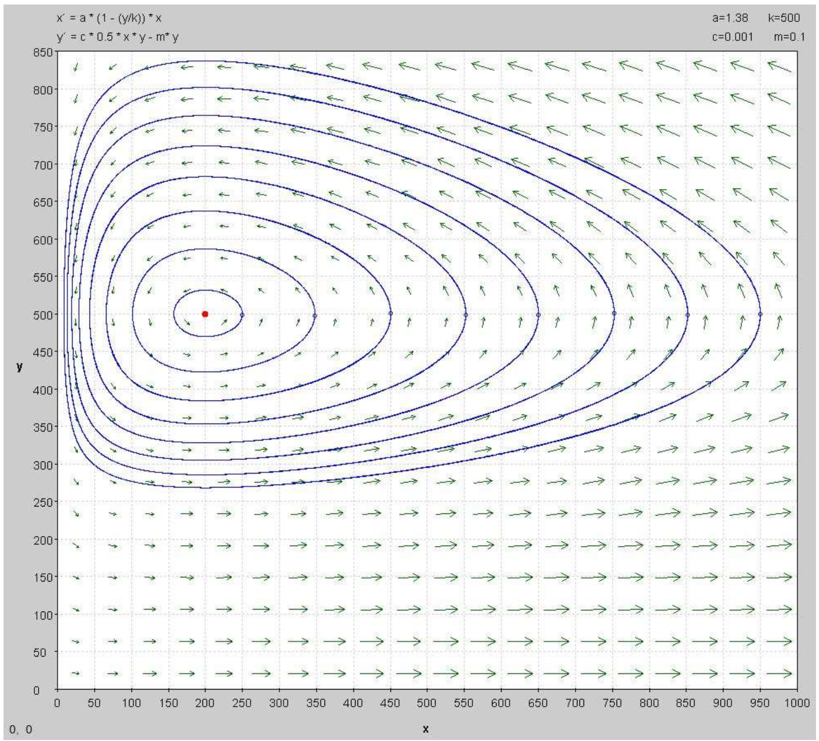
From the graphical analysis, we determine that the behavior of the two populations is a periodic cycle. When the population of the butterflies decrease, it follows that the population of the wasps decrease because there are not enough eggs for them to lay their own eggs in. The decline in wasp population allows the butterfly population to grow as fewer eggs are being parasitized by the wasps. The growing butterfly population provides more opportunity for the wasp population to lay eggs. This causes the butterfly population to decrease, starting the cycle over again.

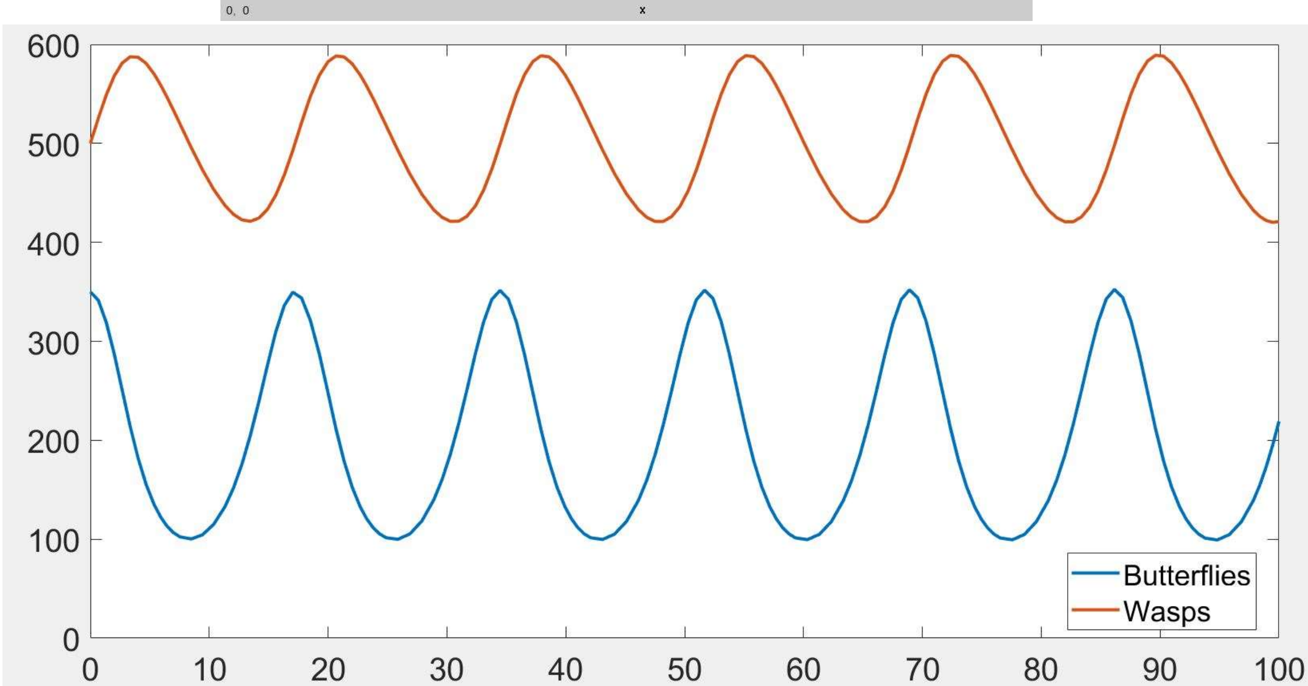
After examining the graphical analysis, we decided to try and estimate some actual values for our parameters. Starting with a, the growth rate of butterflies, we found that one butterfly lays two lots of about 30 eggs in a lifetime. Because only female butterflies lay eggs, we can average this out as one butterfly laying 30 eggs in a lifetime. The average lifespan of a butterfly is three weeks. So, the birth rate of butterflies is $\frac{30\ eggs}{21\ days}$. We accounted for both the birth and death rate in our growth rate parameter. For the death rate, we found one butterfly dies every 21 days. Taking the birth rate minus our death rate, our growth rate is $\frac{29\ eggs}{21\ days} \approx 1.38$. In a similar fashion, we found the death rate of wasps, m, to be $\frac{1\ wasp}{10\ days} = 0.1$.

The other two variables were a little harder to find, so we attempted to come up with some reasonable values. For k, the number of wasps it would take to parasitize all the butterfly eggs, we guessed to be 500, thinking there were maybe 500 wasps in an ecosystem at a given time. To determine the last parameter c, we used a numerical simulation until we had a shape that looked like what we expected where $c = 0.001$.

In conclusion, the two populations are dependent upon one another and have a sustainable relationship given both populations are healthy. The closer these populations start to the coexistence equilibrium, $(200, 500)$ given our parameters, the more stable the populations are. That is, there are less fluctuations in the populations. As the initial condition gets farther from this equilibrium, it is clear that the butterfly population can become dangerously low. These ideas are visible in the graph below.

Phase Plane and Numerical Simulation





Sources

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