Drake University Team 3
Mentor: Terrance Pendleton

Statement of Problem: Model the interactions of the male and female P. brassicae as well as parasitic wasps.

The P. brassicae females release chemical signals which then promote anti-aphrodisiacs to be released by the male population. The chemical signals from females lead to parasitics wasps consuming the eggs laid by P. brassicae. The anti-aphrodisiacs support more efficient mating and egg-placement of P. brassicae but also increase the likelihood that the eggs will be eaten by parasitic wasp larvae. From this knowledge we have five assumption:

(1) The amount of anti-aphrodisiacs present is proportional to the interactions between the male and female P. brassicae interactions
(2) The amount of viable eggs laid by P. brassicae is proportional to the amount of anti-aphrodisiacs present
(3) P. brassicae eggs are susceptible to parasitic wasps proportional to the amount of anti-aphrodisiacs
(4) P. brassicae eggs have equal likelihood of being male or female
(5) The P. brassicae are essential for parasitic wasp survival

From these assumptions we have formed the following system of equations:

$$F'(t) = \frac{(caM(t)^2F(t)^2 - cM(t)F(t)W(t))}{2} - dF(t)$$

$$M'(t) = \frac{(caM(t)^2F(t)^2 - cM(t)F(t)W(t))}{2} - eM(t)$$

$$W'(t) = bW(t) + cM(t)F(t)W(t) - gW(t)$$

$$where\ a, b, c, d, e, g > 0\ and\ g > b$$

In this model, we have three differential equations F'(t), M'(t), and W'(t) representing female P. brassicae, male P. brassicae, and parasitic wasp populations respectively. Parameter c represents the proportion of anti-aphrodisiacs present while parameter a represents the survival rate of the P.brassicae eggs. Parameters d, e, and g are the death rates for their respective population. Finally, parameter b is the base growth rate for the parasitic wasps that is not dependent on P. brassicae. Our time parameter, t, is in weeks.

To begin our long-term population trend analysis, we found three equilibrium solutions:
(1) (W(t) = 0, F(t) = 0, M(t) = 0)
(2) (W(t) = 0, F(t) = $\frac{2d}{ca(\frac{2d^2}{cae})^{2/3}}$, M(t) = $\sqrt[3]{\frac{2d^2}{cae}}$)
(3) (W(t) = $\frac{g-b}{e}\sqrt{\frac{de}{c(g-b)}}$, F(t) = $\frac{e}{c}\sqrt{\frac{c(g-b)}{de}}$, M(t) = $\frac{a(g-b)}{2ec} - 2\sqrt{\frac{de}{c(g-b)}}$)

Due to the computationally extensive nature of our third solution and in an interest of time, we decided to focus our analysis on the first two equilibrium solutions. Using a Jacobian matrix, we were able to find the eigenvalues for our first two equilibrium solutions:

Equilibrium Solution (1): $\lambda 1 = -e$, $\lambda 2 = -d$, $\lambda 3 = b-g$

Equilibrium Solution (2): $\lambda 1 = \frac{e+d}{2} + \frac{1}{2}\sqrt{e^2 + d^2 + 14ed}$, $\lambda 2 = \frac{e+d}{2} - \frac{1}{2}\sqrt{e^2 + d^2 + 14ed}$,

$$\lambda 3 = (b-g) + c^3\sqrt{\frac{4de}{ca^2}}$$

Since we are working with differential equations, in order for an equilibrium solution to be stable all of the eigenvalues of its Jacobian matrix must be strictly less than 0. Due to our initial restraints set on our parameters described above, we can see that the first equilibrium solution will always be stable. This is because g > b, e and d > 0. Similarly, we are able to observe that the second equilibrium solution can never be stable. When looking at the first eigenvalue, we can note that all of our parameters are strictly greater than 0. Because of this, the first eigenvalue will also always be greater than 0 and thus the whole equilibrium solution is unstable.

As we began testing possible parameters for our equations, we noticed that the trend of populations over time most likely would result in complete extinction of both species or extinction of just the wasp species resulting in exponential growth of butterflies. We were unable to find a suitable set of parameters that led to long term co-existence between the butterflies and the wasps.

With starting population M(0) = 11,000, F(0) = 8,000, W(0) = 15,000 and parameters a = 0.0001, b = 0.04, c = 4e-10, d = 0.0037, e = 0.002, g = 0.085 we can observe a scenario where wasps die out but the butterflies survive and grow exponentially. Figure 1 shows the population trend over 700 time steps. We used a flooring function so that once the wasp population was less than 1, it would count as extinct. From Figure 1 we can see that there were not enough butterflies to sustain the wasp population. Figure 2 shows a 2D view of our 3D vector field. The x-axis is male butterflies and the y-axis is female butterflies. From this vector field, we can see that the butterfly populations are likely to go extinct or shoot up to infinity. Figure 3 shows the 3D vector field.

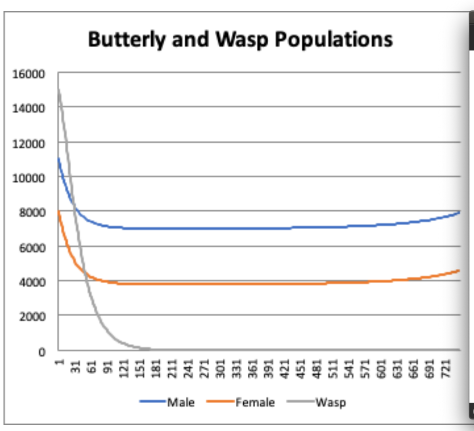

Figure 1

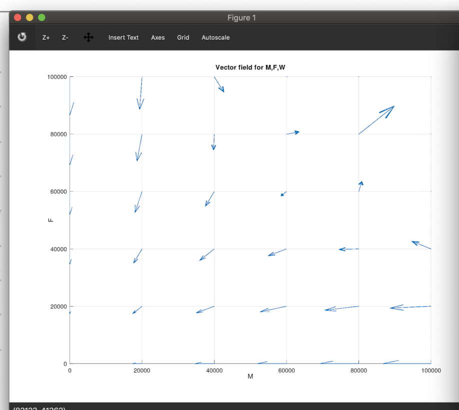

Figure 2

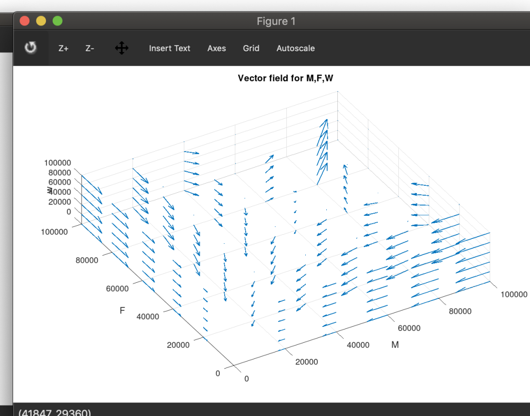

Figure 3

Conclusion:

Our model was able to generate populations for wasps as well as female and male butterflies over time. Without another way to increase the growth rate of wasps, butterflies are likely not sufficient enough to keep the population alive. Similarly, if there are too many wasps and not enough anti-aphrodisiacs, the butterfly population will not be able to reproduce quickly enough to survive.

Sources:

Cell Press. "A Boy For Every Girl? Not Even Close: Scientists Trace Evolution Of Butterflies Infected With Deadly Bacteria." ScienceDaily. ScienceDaily, 11 September 2009. <www.sciencedaily.com/releases/2009/09/090910121801.htm>

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