Executive Summary; Problem C: Chemical Espionage

Lakeland Community College; Elliott Yoon, Dane Miller, Cameron Byrne; coach: Paul Zachlin

I. GENETIC VARIATION IN MALE AND FEMALE P. BRASSICAE AND ITS INFLUENCE ON PARASITIC WASPS

To start, assume there is genetic variation in the population, therefore some butterflies produce anti-aphrodisiac while other do not. Let B_{xx} denote the female butterfly population, B_{YA} denote the anti-aphrodisiac producing male population, and B_{Ya} denote the non-anti-aphrodisiac producing male population. Since only males produce the anti-aphrodisiac, it can be assumed to be a genetically Y-linked trait.

	X	Y_A
X	XX	XY_A
X	XX	XY_A

	X	Y_a
X	XX	XY_a
X	XX	XY_a

Because of the above Punnett squares, the probability of each genotype being produced given one random reproduction is :

$$P(XX) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} + \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \tag{1}$$

$$P(XY_A) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \tag{2}$$

$$P(XY_a) = \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \tag{3}$$

Note that these are the probabilities without any risk of wasps and assuming all of the eggs survive. Accounting for the wasps and natural causes of egg death, the probabilities can be modeled more accurately as:

$$\begin{aligned} P(XX) = &\frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \cdot ((1 - P(\text{wasp})) \\ &\cdot P(M_A)) + \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a) \end{aligned} \tag{4}$$

$$P(XY_A) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \cdot ((1 - P(\text{wasp})) \cdot P(M_A) \tag{5}$$

$$P(XY_a) = \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a) \tag{6}$$

where $P(\text{wasp})$ is the probability of a wasp destroying the egg, $P(M_A)$ is the probability that an egg with the anti-aphrodisiac benefits will mature to adulthood, and $P(M_a)$ is the probability that an egg without anti-aphrodisiac benefits will mature to adulthood.

So, given n randomly reproduced eggs are laid, $n \cdot P(XX)$ = number of females maturing to adulthood, $n \cdot P(XY_A)$ = number of anti-aphrodisiac producing males maturing to adulthood, and $n \cdot P(XY_a)$ = number of non-anti-aphrodisiac producing males maturing to adulthood.

Given γ is hunting efficiency W is the wasp population, and α is birth rate of the butterflies, we can model the change in wasp population using a Lotka-Volterra model as such,

$$\frac{dW}{dt} = \gamma W \cdot B_{\text{YA}} - \alpha W \tag{7}$$

Assuming the wasps and the butterflies are the only predators and prey, respectively, in the system, the rate at which eggs are destroyed is proportional to the wasp population times the anti-aphrodisiac producing population, as the wasps use the anti-aphrodisiacs to find the butterfly eggs and thus do not prey on the non-anti-aphrodisiac population.

Now, letting E be the number of eggs destroyed, μ being some proportionality constant, and k being the average eggs laid, we have that

$$E' = \mu W (B_{YA} \cdot k) \tag{8}$$

$$E = k\mu \int W \cdot B_{YA} \; dt \tag{9}$$

where $\int W_{\text{YA}} \; dt$ can be trivially approximated using Euler's method, left as an exercise to the reader. Then, knowing the probability of a wasp finding a butterfly egg, $P(\text{wasp})$ is the number of eggs destroyed divided by the product of the total anti-aphrodisiac positive population and the average number of eggs laid per wasp $\frac{E}{k \cdot B_{YA}}$,

$$P(\text{wasp}) = \frac{\cancel{k}\mu \int W \cdot B_{YA} \; dt}{\cancel{k} \cdot B_{YA}} \tag{10}$$

Then, then the three probabilities $P(XX)$, $P(XY_A)$, and $P(XY_a)$ become

$$\begin{aligned} P(XX) = &\frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \cdot ((1 - \frac{k\mu \int W \cdot B_{YA} \; dt}{k \cdot B_{YA}})) \\ &\cdot P(M_A)) + \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a) \end{aligned} \tag{11}$$

$$\begin{aligned} P(XY_A) = &\frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \\ &\cdot ((1 - \frac{k\mu \int W \cdot B_{YA} \; dt}{k \cdot B_{YA}})) \cdot P(M_A) \end{aligned} \tag{12}$$

$$P(XY_a) = \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a) \tag{13}$$

Finally, we solve for equations to model the total butterfly population. Noting that the rate of female butterfly population change B'_{XX} is female birth rate B'_{BXX} minus female death rate B'_{DXX}, we have that $B'_{XX} = B'_{BXX} - B'_{DXX}$. Since for n eggs laid and B_{BXX} female births, $n \cdot P(XX) = B_{BXX}$ and likewise $n = \frac{B_{BXX}}{P(XX)}$. Taking the derivative with respect to time,

$$n' = \frac{P(XX) \cdot B'_{BXX} - B_{BXX} \cdot P'(XX)}{P^2(XX)} \tag{14}$$

Knowing n' = birth rate proportion constant k times total butterfly population B_T, we have that

$$k \cdot B_T = \frac{P(XX) \cdot B'_{BXX} - B_{BXX} \cdot P'(XX)}{P^2(XX)} \tag{15}$$

and equivalently

$$k \cdot B_T = \frac{P(XY_A) \cdot B'_{BXY_A} - B_{BXY_A} \cdot P'(XY_A)}{P^2(XY_A)} \tag{16}$$

$$k \cdot B_T = \frac{P(XY_a) \cdot B'_{BXY_a} - B_{BXY_a} \cdot P'(XY_a)}{P^2(XY_a)} \tag{17}$$

We can say $B'_{DXX} = d \cdot B_{XX}$, where d is the death rate proportion constant, and therefore $B'_{BXX} = B'_{XX} + d \cdot B_{XX}$. Remember $B_{BXX} = n \cdot P(XX)$, and we can substitute the two as follows:

$$k \cdot B_T = \frac{P(XX) \cdot (B'_{XX} + d \cdot B_{XX}) - (n \cdot P(XX) \cdot P'_{XX})}{P^2(XX)} \tag{18}$$

Since $n' = k \cdot B_T \Rightarrow n = k \int B_T dt$

$$k \cdot B_T = \frac{P(XX) \cdot (B'_{XX} + d \cdot B_{XX})}{P^2(XX)} - \frac{((k \int B_T dt) \cdot P(XX) \cdot P'_{XX})}{P^2(XX)} \tag{19}$$

$$(k \cdot B_T \cdot P^2(XX)) - P(XX) \cdot (B'_{XX} + d \cdot B_{XX}) = - (k \int B_T dt) \cdot P(XX) \cdot P'_{XX} \tag{20}$$

$$\frac{k \cdot B_T \cdot P(XX) - B'_{XX} - d \cdot B_{XX}}{-k \cdot P'(XX)} = \int B_T dt \tag{21}$$

Differentiating both sides yields:

$$B_T = \frac{-k \cdot P'(XX)(K \cdot B_T \cdot P'(XX) + k \cdot B'_T \cdot P(XX)}{(k \cdot P'(XX))^2} - \frac{(-k \cdot P'(XX))(-B''_{XX} - d \cdot B'_{XX})}{} + \frac{k \cdot P''(XX)(k \cdot B_T \cdot P(XX) - B'_{XX} - d \cdot B_{XX})}{(k \cdot P'(XX))^2} \tag{22}$$

and equivalently changing $P(XX)$ and $P'(XX)$ to $P(XY_A)$,$P'(XY_A)$ and $P(XY_a)$,$P'(XY_a)$, respectively, to yield two more equations modeling the butterfly populations, as in equations (15-17), we have a system of differential equations, albeit optimistic and assuming no other competition, to model the butterfly populations.

II. P. BRASSICAE AND WASP POPULATION DYNAMICS VIA LOTKA-VOLTERRA

A well-known model for predator-prey dynamics is the Lotka–Volterra model, which is a pair of first-order differential equations, often represented as:

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

Where α represents per capita birth rate of prey, β represents per capita prey mortality rate due to predation, δ represents predator hunting efficiency, and γ represents per capita mortality rate of predators.
A variant of the model was used previously to model the change in wasp populations. Using Lotka-Volterra to model the wasp W and butterfly anti-aphrodisiac positive B_{YA} populations, with $\alpha = 0.1$, $\beta = 0.01$, $\delta = 0.1$, and $\gamma = 0.002$, we have

$$\frac{dB_{\mathrm{YA}}}{dt} = \alpha B_{\mathrm{YA}} - \beta W \cdot B_{\mathrm{YA}} \tag{23}$$

$$\frac{dW}{dt} = \delta W \cdot B_{\mathrm{YA}} - \gamma W \tag{24}$$

which we can then solve using Runga-Kutta and visualize the population dynamics between P. Brassicae butterflies and the wasps.

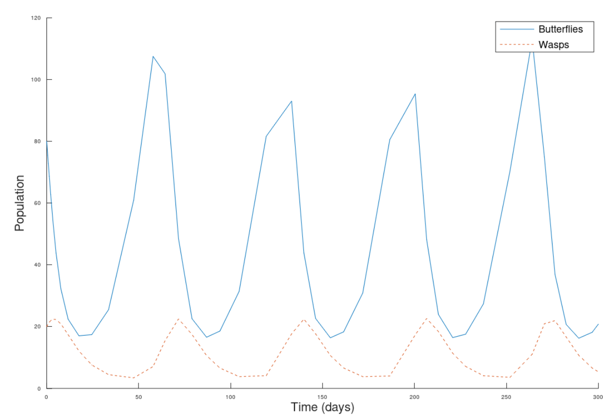

FIG. 1. Population model between wasps and butterflies