Problem C: Chemical Espionage Response

Jamal Lawson, Damodar Dahal, Eraj Khatiwada, Edgar Reyes (Coach)

SCUDEM IV 2019

In response to this question we allow $x_1(t)$ to be the number of female butterflies, $x_2(t)$ the number of male butterflies, $x_3(t)$ the number of parasitic wasps of either type, and $x_4(t)$ to be the number of butterfly eggs, hijacked or not, at a time $t \geq 0$. Then we consider the system of ordinary differential equations where $k_i \geq 0$ for every $i = 1, 2, 3, 4, 5, 6, 7, 8$ and $a_0, b_0, c_0 > 0$:

$$
\begin{aligned}
\dot{x}_1(t) &= -k_1 x_1(t) + k_2 x_4(t) & x_1(0) &= a_0 \\
\dot{x}_2(t) &= -k_3 x_2(t) + k_4 x_4(t) & x_2(0) &= b_0 \\
\dot{x}_3(t) &= -k_5 x_3(t) + k_6 x_4(t) & x_3(0) &= c_0 \\
\dot{x}_4(t) &= -k_7 x_4(t) + k_8 x_1(t) x_2(t) & x_4(0) &= 0.
\end{aligned}
\tag{1}
$$

In each equation, we note that we in each species and gender an insect should die off at some point (thus, the $-k_i x_j(t)$ term) and we expect some eggs to hatch as an insect (thus, the $+k_m x_4(t)$ terms). Also, in the fourth equation of the autonomous system, we make special note of the mixed term $x_1(t)x_2(t)$. We would expect there to be no eggs without any interaction of between the male and femal butterflies. Without any such intersections, this term is simply 0. To find an equilbrium solution (1), we allow $\dot{x}_j = 0$ for $j = 1, 2, 3, 4$, which yields an equilibrium solution of $(x_1(t), x_2(t), x_3(t), x_4(t)) = (a_0, b_0, c_0, 0)$.

Because this system is nonlinear and involves so many dependent variables, it is not exactly like a Lotka-Volterra system, and we do not analyze it as such. Thus, we attempt look at a few simpler cases.

<u>**Case 1**</u>: $x_2(t) = 0\ \forall t$: In this case, there are no male butterflies in the vicinity to respond to the aphrodisiac of the females. Thus, the release no anti-aphrodisiac; and we expect no eggs to be produced, and a dying off of the female butterflies as well as the wasps. Indeed this is the case. We look

first at the equation for the eggs. Since $\dot{x}_4(t) = -k_7x_4(t) + k_8x_1(t)x_2(t)$, with $x_2(t) = 0$, we see that $\dot{x}_4(t) = -k_7x_4(t)$. This equation has as its general solution $x_4(t) = x_4(0)e^{-k_7t} = 0$. Thus, there are no eggs produced. With this, then, the x_4 terms in the equations for female butterflies and parasitic wasps are 0, and the solutions to these reduced equations are $x_1(t) = a_0e^{-k_1t}$ and $x_3 = c_0e^{-k_5t}$. Thus, all species simply die out in the absence of males.

Note that we may generalize this case to have the mixed term $x_1(t)x_2(t) = 0$. This simply means that either no male or no female butterflies are present and the number of eggs is simply zero always still, which reduces each population to simply dying out. This, of course, makes sense physically since a species that does not reproduce should simply disappear.

Case 2: $x_3(t) = 0 \ \forall t$: In this case, we assume that no parasitic wasps are present to pick up the chemical anti-aphrodisiac released by the male butterflies to ward off other males. We should expect the population of butterflies to simply grow. With this, then, the system reduces to:

$$\begin{aligned}
\dot{x}_1(t) &= -k_1x_1(t) + k_2x_4(t) & x_1(0) &= a_0 \\
\dot{x}_2(t) &= -k_3x_2(t) + k_4x_4(t) & x_2(0) &= b_0 \\
\dot{x}_4(t) &= -k_7x_4(t) + k_8x_1(t)x_2(t) & x_4(0) &= 0.
\end{aligned} \tag{2}$$

However, since there are no wasps present, all of the eggs that will hatch are either male or female butterflies. Thus, there are some positive real numbers α and β such that $x_4(t) = \alpha x_1(t) + \beta x_2(t)$. Thus, we amy further reduce the system (2) to:

$$\begin{aligned}
\dot{x}_1(t) &= (-k_1 + k_2\alpha)x_1(t) + k_2\beta x_2(t) & x_1(0) &= a_0 \\
\dot{x}_2(t) &= (-k_3 + k_4\beta)x_2(t) + k_4\alpha x_1(t) & x_2(0) &= b_0.
\end{aligned} \tag{3}$$

In matrix form, without the initial conditions immeditely, we rewrite this as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -k_1 + k_2\alpha & k_2\beta \\ k_4\alpha & -k_3 + k_4\beta \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{4}$$

Because we do not have the nature of the trace nor of the determinant of this matrix, this was as far as we came with this simplication without more time. We also could not solve the general system due to its complexity.