SCUDEM IV 2019 - Problem B

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1 Introduction

In our approach, we consider the probe decelerating using its thrusters before touchdown. On touchdown, the probe's impact is absorbed by three legs with springs. Our models will be focused on these mass-spring systems.

2 One Spring System and Landing

Consider a space probe with mass m (kg) approaching an asteroid. The distance of the center of mass to the end of the spring is u (m) where $u = 0$ when in equilibrium (not compressed or stretched). The oscillations of the spring is slowed down by a frictional force described by the dampening coefficient γ and the spring coefficient k describes how stiff the spring is. The probe uses thrusters to decelerate at a rate of a (m/s/s) to a velocity of v_0 before touching down on the asteroid.

The probe has three legs but we treat these legs as one vertical leg to simplify calculations.

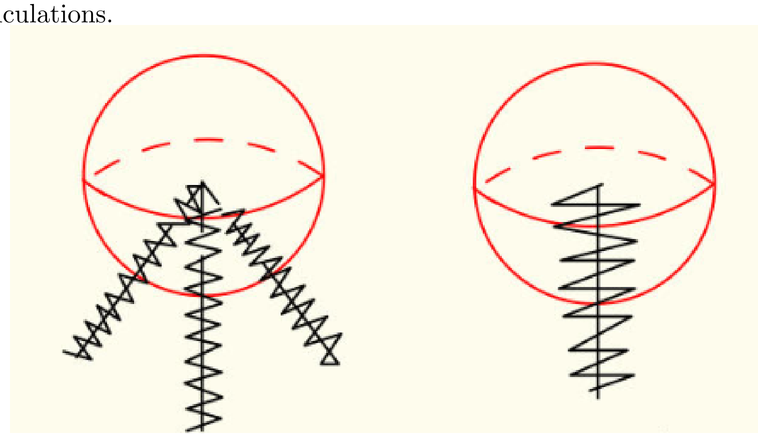

Figure 1: Actual probe and simplified probe.

As the probe lands, the springs are going to absorb some of the force from landing and, if the force is strong enough, the probe will bounce.

As a spring is compressed, it pushes back with a force of

$$F_s = m\frac{d^2u}{dt^2} = -ku - \gamma\frac{du}{dt}. \quad (1)$$

When the spring is compressed, u is negative and force is positive. When the spring is stretched, u is positive and force is negative.

The asteroid is accelerating the probe's decent at a rate of g and the thrusters are deaccelerating the probe at a rate of a. When the probe first touches down on the asteroid, it hits the surface with a force of $F_L = mg + ma$. In this case, g is positive and a is negative.

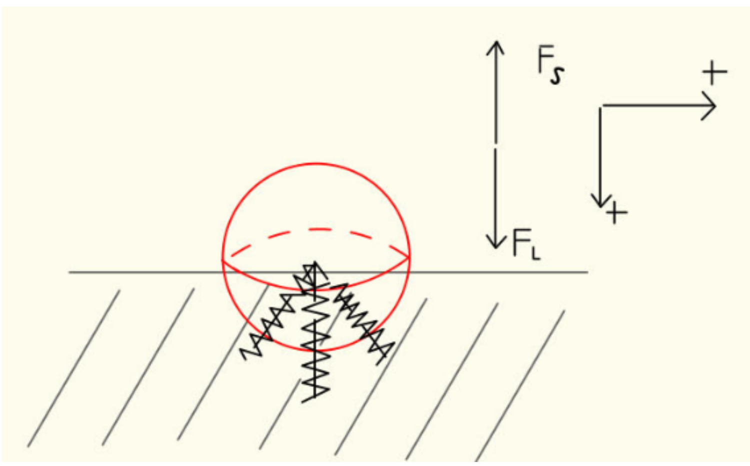

Figure 2: Probe landing on asteroid

There are three cases of damping: overdamping (no oscillation), underdamping (oscillation occurs), and critical damping (no oscillation but on the verge of oscillation). We chose to underdamp our spring since it should absorb the most force and make the landing less jarring.

To find the oscillation of the spring we solve for u. Equation 1 can be rewritten as

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = 0.$$

The the characteristic equation is

$$mr^2 + \gamma r + k = 0$$

which give the characteristic roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

For the spring to be underdamped we want $\gamma^2 < 4mk$ so the complex roots are

$$r = -\frac{-\gamma}{2m} \pm \omega i$$

where

$$\omega = \frac{\sqrt{2mk - \gamma^2}}{2m}$$

then the general solution is

$$u = e^{\frac{-\gamma t}{2m}}(c_1 \cos \omega t + c_2 \sin \omega t) \tag{2}$$

where c_1, c_2 are constants.

In this equation, the sine and cosine functions give oscillation and the negative exponent on e dominates, eventually making oscillation reach close to zero.

We then assume the probe is descending in the vertical direction.

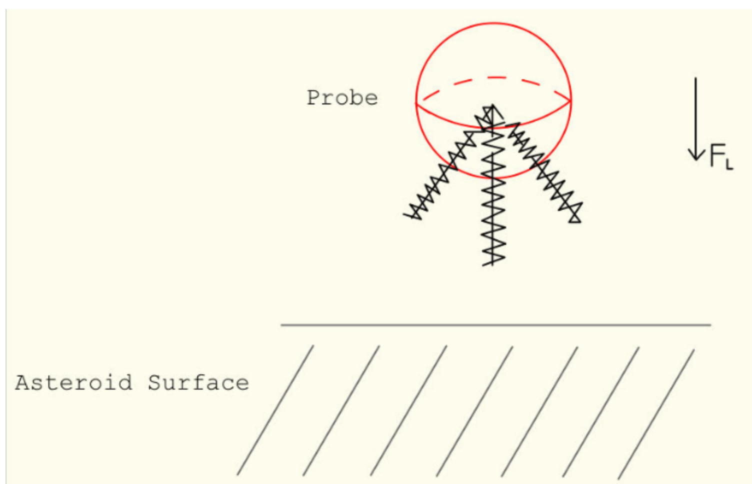

Figure 3: Probe approaching asteroid

Let x (m) be the vertical displacement of the probe approaching the asteroid. The probe travels towards the asteroid with an initial velocity of x_0' and over a distance of Δx with an acceleration of $g + a$. The velocity of the probe as it lands on the surface of the asteroid is

$$x' = \sqrt{(x_0')^2 + 2(g + a)\Delta x}.$$

Then the moment the probe lands, the spring compresses at a velocity of $u' = x'$ with initial spring displacement of $u_i = 0$. This give the initial conditions $u'(0) = x'$ and $u(0) = 0$. Using these initial conditions to solve the general equation (2) gives

$$\begin{aligned} u(0) = e^0(c_1 \cos 0 + c_2 \sin 0) = 0 \\ u(0) = c_1 \cos 0 = 0 \end{aligned}$$

which implies $c_1 = 0$.

Differentiating the general solution gives

$$u'(0) = c_2 \omega \cos(0) = x'$$

which implies $c_1 = \frac{x'}{\omega}$.

Substituting the values of c_1 and c_2 into the general solution gives the oscillation of the spring when the probe lands on the asteroid

$$u(t) = e^{\frac{-\gamma t}{2m}} x' \sin \omega t. \tag{3}$$

Arbitrary parameter values were selected and used to graph the movement of the springs.

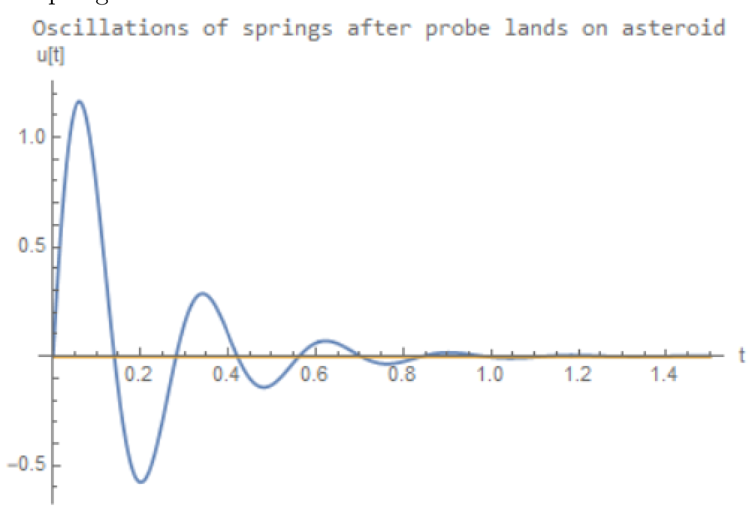

Figure 4: Oscillation of probe

2.1 Additional issues to consider in the future

- If the probe is descending at an angle or is landing on the side of a hill then not all of the landing force will go into the spring, in which case the landing angle will be needed to determine how much of the landing force is going into the springs.
- The three legs of the probe do not truly act as one vertical leg. Since the legs are at an angle to the surface of the asteroid, not all of the landing force will go into the springs.

Let F_N be the normal force. Only the part of the normal force will be pushed into the spring so

$$F_s = F_N cos\theta$$

$$\gamma \frac{du}{dt} + ku = (mg + ma)cos\theta$$

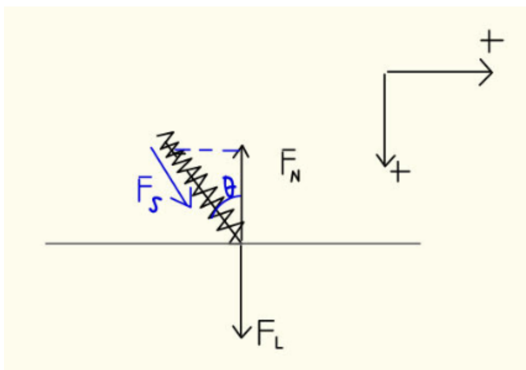

Figure 5: Acting forces

- Because the surface is very uneven, one springs could land before the others and it may begin to compress and expand sooner which would tilt the probe and increase or decrease the amount of force the other legs land with. We might be able to assume that we will choose a landing location and speed where this will not be a significant problem but if now, the equation we would be may be

$$m\frac{d^2u}{dt^2} + \gamma \frac{du}{dt} + ku = F(t)$$

3 Hopping to Predetermined Point

In this problem, we assume we use the springs on the leg to hop to a predetermined point. The springs will compress, some more than others, then the spring will release to 'hop' and move the probe around in the chosen direction on the asteroid. We must be careful with the velocity the spring will be released at, since if we exceed the escape velocity of the asteroid, we can not come back as stated in the sixth section.

We say x is the displacement of the probe and x' is it's velocity. After the springs have been fully compressed, they will expand, propelling the probe in a chosen direction. The displacement of the probe will be equal to the the distance the springs moved until they returned to equilibrium so $\Delta x = \Delta u$. So we have the equation

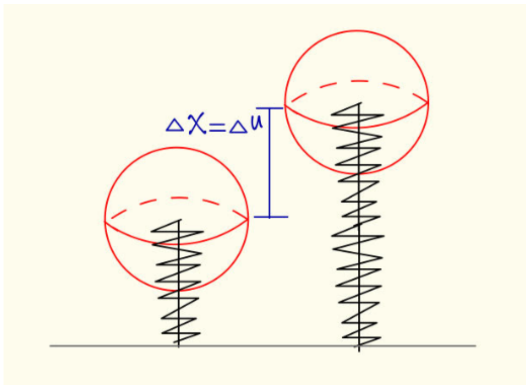

Figure 6: $\Delta x = \Delta u$

$$x_i' = \sqrt{(x_{i-1}')^2 + 2a\Delta u}. \tag{4}$$

The acceleration a in the equation can be obtained by using equation (1) for F_s from the previous section. The initial velocity of the spring when the probe lets the spring loose is zero so the equation is reduced to $F_s = -ku$. Next, we let

$$F_s = ma$$

So

$$a = F_s/m$$

Hence

$$a = -ku/m.$$

The velocity at the end of the first hop x_1' is given by $\sqrt{2a\Delta u}$ since $x_0'$is zero. To find displacement, we must first define x_{x_i}' and x_{y_i}'. These represent the velocity in the x and y direction, respectively. So x_{x_i}' is given by

$$x_{x_i}' = x_i' cos\theta$$

and x_{y_i}' is given by

$$x_{y_i}' = x_i' sin\theta + gt.$$

To find the displacement, we let $x_i' = 0$ at a certain time t_{top}. Then,

$$x_{y_i}' = +gt$$

$$t_{top} = \frac{x_{y_i}'}{g}.$$

The time travelled will be twice the t_{top} so

$$t_{bottom} = \frac{2x_{y_i}'}{g}.$$

Hence the total displacement is $x'_{x_i} \times t_{bottom}$ and that gives

$$\Delta x_i = \frac{2x'_{x_i} x'_{y_i}}{g}.$$

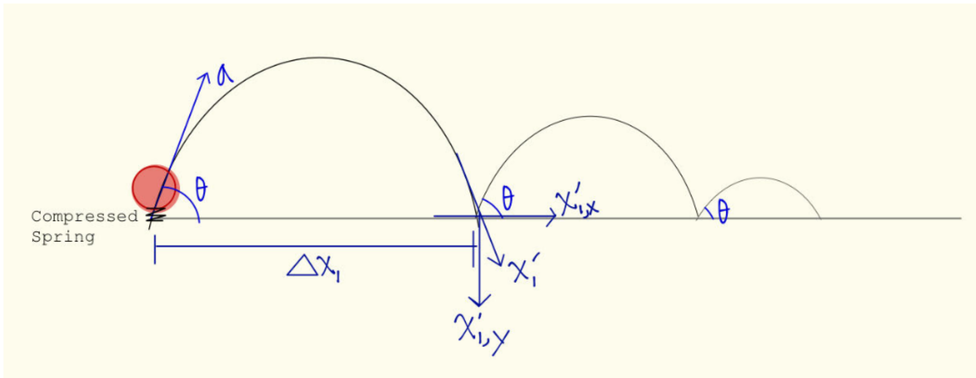

Figure 7: Probe hopping

We will repeat this process by plugging in the obtained velocity and displacement into equation (4) and go on until the displacement is small enough, it would not affect the landing position by that point in time. Some constraints are that x_i must never exceed the escape velocity, and θ should be close to $\pi/2$ for maximum distance and shorter hops as given in the graph below.

3.1 Additional issues to consider in the future

After hopping, the force the probe lands with may not be powerful enough to push the probe into the air again. We should find how for the probe can travel before the spring will absorb the remaining force from landing.

4 Jumping Into Ravine

The probe must have a stable source as its hopping point, and the consecutive hops must follow since a change in direction may result in a fatal crash. It must calculate the exact distance so it doesn't go too far or too short, and it also cannot exceed the escape velocity.

If the probe is sitting at the top of a ravine and jumps in the x direction into the ravine with a velocity of $x'_{i,x}$ and $x'_{i,y} = 0$. The probe lands in the ravine with the y-component of velocity being

$$x'_y = \sqrt{(x'_{i,y})^2 + 2g\Delta x_y} = \sqrt{2g\Delta x_y}$$

This value of x'_y can replace x' in equation 3 to fins the oscillation of the springs.

To jump out of the ravine, the probe can hop out as described in section three.

4.1 Details to consider in the future

To simplify calculations and save time, we chose the equations so that the probe only jumps in the x direction to jump off the edge. In reality, the probe will likely have to jump at an angle to reduce friction and avoid hitting rocks from the uneven terrain.

5 Hopping Along a Steep Cliff

The probe obviously cannot fall into the cliff since it would likely not be able to come back due to the unknown ground formations which will not support hopping. The probe must start its hop from relatively stable ground and not exceed the escape velocity.

5.1 Additional issues to consider in the future

Ideally, the probe will not hop with a force strong enough for the probe to bounce a second time. If the probe bounced a second time there is a risk of it hopping too far and falling off the cliff. To find the maximum usable force, we need to find a force such that the springs absorb the force from landing on the asteroid again. This may look like a combination of sections one and two.

6 Asteroid Dimensions

We conclude the asteroid dimensions cannot exceed the conditions given in the escape velocity, G and r. Our probe cannot exceed the escape velocity since it would fly out into space and never be able to return. The asteroid will have to have a large mass the create enough gravity to keep the probe on the surface as it hops.

The escape velocity of a body with mass M and radius r is given by;

$$v_e = \sqrt{\frac{2GM}{r}}.$$

Where $G \approx 6.67 \times 10^{-11}\, \mathrm{m^3 kg^{-1} s^{-2}}$ is the universal gravitational constant.

6.1 Additional issues to consider in the future

The size of the asteroid the probe can land on may also be affected by how much the probe bounces across the asteroid when it initially lands. This would be especially important when the probe lands at an angle, on a hill, or on a very uneven surface. We should figure out the distance it takes the probe to come to a stop and base the allowable dimensions on that number.

7 References

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