Problem C: Chemical Espionage

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To start, let’s assume there is genetic variation in the population of butterflies: either they use the anti-aphrodisiac or they don’t use it.

Since only males display the anti-aphrodisiac production trait, we can assume the trait is Y-linked.

Let B_{XX} denote the female butterfly population

Let B_{YA} denote the male butterfly population that produces the anti-aphrodisiac

Let B_{Ya} denote the male butterfly population that does not produce the anti-aphrodisiac

Because the trait is Y-linked, there are only two reproduction possibilities, whose offspring probabilities are given by the punnett squares below

	X	Y_A
X	XX	XY_A
X	XX	XY_A

	X	Y_a
X	XX	XY_a
X	XX	XY_a

Therefore, assuming random reproduction, the odds of any one egg turning into each of the genotypes is as follows:

$$P(XX) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} + \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}}$$

$$P(XY_A) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}}$$

$$P(XY_a) = \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}}$$

Accounting for natural causes of death before the butterflies mature, these would be the new probabilities:

$$P(XX) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \cdot ((1 - P(\text{wasp})) \cdot P(M_A)) + \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a)$$

$$P(XY_A) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \cdot ((1 - P(\text{wasp})) \cdot P(M_A)$$

$$P(XY_a) = \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a)$$

Given that ‘n’ eggs are randomly produced, There should be:

nP(xx) females reaching adulthood,
nP(xyA) anti-A. producing males reaching adulthood, and
nP(xya) non-anti-A. Producing males reaching adulthood

Now let's explore what P(wasp) is. We're going to borrow logic from the Lotka Volterra model, which states that:

$$\frac{dx}{dt} = \alpha x - \beta x y$$

$$\frac{dy}{dt} = \delta x y - \gamma y$$

Using this to model the wasp population assumes that the only way for the wasps to reproduce is by laying their eggs in the butterfly eggs, but we’re going with it.

Applying that model to our system yields

$$\frac{dW}{dt} = \gamma W \cdot B_{YA} - \alpha W$$

Now, Let E = the number of eggs destroyed by the wasps

$$E' = \mu W(B_{YA} \cdot k)$$

$$E = k\mu \int W \cdot B_{YA}\, dt$$

Since P(wasp) = $\dfrac{E}{k \cdot B_{Y\,A}}$

Substituting E from the equation before yields:

$$P(\text{wasp}) = \frac{\cancel{k}\mu \int W \cdot B_{Y\,A}\, dt}{\cancel{k} \cdot B_{Y\,A}}$$

This causes the probabilities to be as follows:

$$P(XX) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \cdot ((1 - \frac{k\mu \int W \cdot B_{YA}\, dt}{k \cdot B_{YA}})) \cdot P(M_A)) + \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a)$$

$$P(XY_a) = \frac{0.5 \cdot B_{Ya}}{B_{YA} + B_{Ya}} \cdot P(M_a)$$

$$P(XY_A) = \frac{0.5 \cdot B_{YA}}{B_{YA} + B_{Ya}} \cdot ((1 - \frac{k\mu \int W \cdot B_{YA}\, dt}{k \cdot B_{YA}})) \cdot P(M_A)$$

The rate of change of the female population is equal to the birth rate minus the death rate. In other words,

$$B'_{XX} = B'_{BXX} - B'_{DXX}$$

The number of females that reach maturity B_{BXX} is equal to the number of eggs laid, n, times P(xx)

$$n \cdot P(XX) = B_{BXX}$$

$$n' = \frac{P(XX) \cdot B'_{BXX} - B_{BXX} \cdot P'(XX)}{P^2(XX)}$$

Now, lets assumed the rate of egg laying is proportional to the total population of butterflies.

So, $$n' = k \cdot B_T$$

Combining the equations,

$$k \cdot B_T = \frac{P(XX) \cdot B'_{BXX} - B_{BXX} \cdot P'(XX)}{P^2(XX)}$$

And following a similar process for males yields:

$$k \cdot B_T = \frac{P(XY_A) \cdot B'_{BXY_A} - B_{BXY_A} \cdot P'(XY_A)}{P^2(XY_A)}$$

$$k \cdot B_T = \frac{P(XY_a) \cdot B'_{BXY_a} - B_{BXY_a} \cdot P'(XY_a)}{P^2(XY_a)}$$

These equations have the birth rate involved, and we want to try to get everything in terms of the populations and their derivatives.

Doing this requires some clever substitution:

We can say

$$B'_{DXX} = d \cdot B_{XX}$$

Therefore,

$$B'_{BXX} = B'_{XX} + d \cdot B_{XX}$$

$$B_{BXX} = n \cdot P(XX)$$

Plugging these into the equation from before,

$$k \cdot B_T = \frac{P(XX) \cdot (B'_{XX} + d \cdot B_{XX}) - (n \cdot P(XX) \cdot P'_{XX})}{P^2(XX)}$$

Since $n' = k \cdot B_T \Rightarrow n = k \int B_T dt$

Plugging that in and doing some algebra to isolate the integral of the total butterfly population yields:

$$\frac{k \cdot B_T \cdot P(XX) - B'_{XX} - d \cdot B_{XX}}{-k \cdot P'(XX)} = \int B_T dt$$

Then, it is possible to differentiate both sides, getting rid of the integral and yielding:

$$B_T = \frac{-k \cdot P'(XX)(K \cdot B_T \cdot P'(XX) + k \cdot B'_T \cdot P(XX)}{(k \cdot P'(XX))^2} - \frac{(-k \cdot P'(XX))(-B''_{XX} - d \cdot B'_{XX})}{} + \frac{k \cdot P''(XX)(k \cdot B_T \cdot P(XX) - B'_{XX} - d \cdot B_{XX})}{(k \cdot P'(XX))^2}$$

Doing the same process for the other two male populations yields two similarly messy equations, but it should be possible to iteratively solve the system of differential equations from there

The prior equations are complicated and it was infeasible to try to get the system into a format in which it could be solved iteratively in a week.

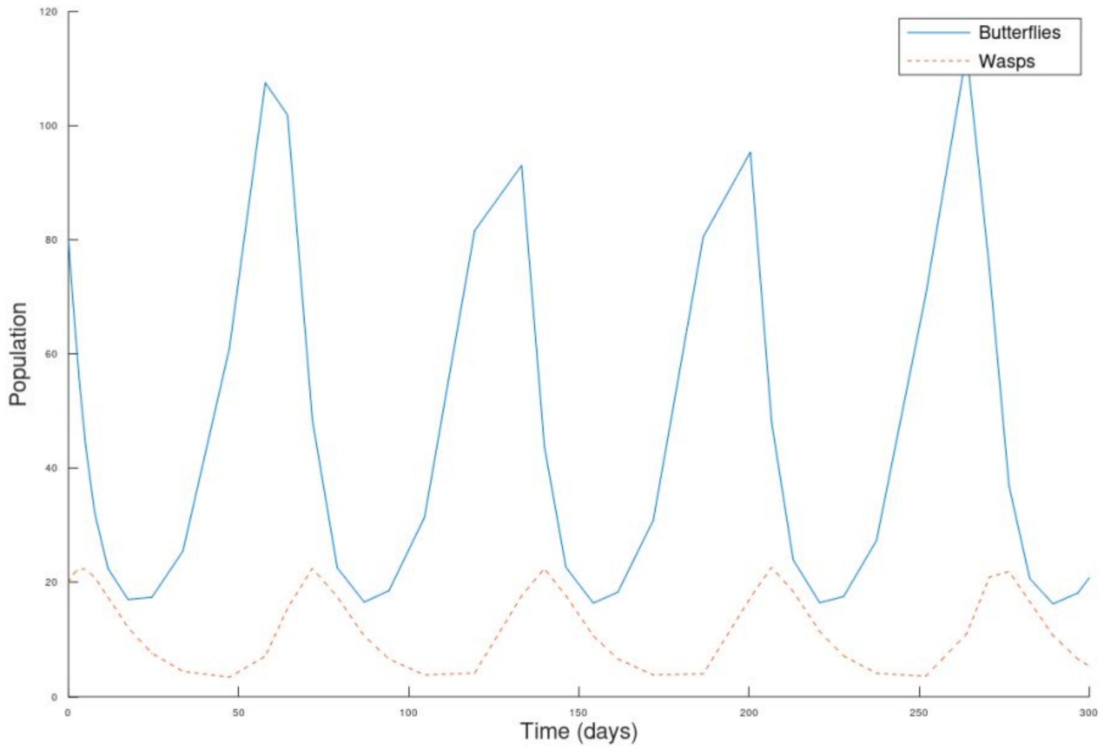
Instead, to get an idea of what the anti-aphrodisiac production population is doing, it is possible to use the Lotka-Volterra model.

This is a simplified version of the model using only the Lotka-Volterra equations.

$$\frac{dB_{\mathrm{YA}}}{dt} = \alpha B_{\mathrm{YA}} - \beta W \cdot B_{\mathrm{YA}}$$

$$\frac{DW}{dt} = \delta W \cdot B_{\mathrm{YA}} - \gamma W$$

Graph of Populations Over Time Using Simplified Model



Added twist no. 2

Given the female P. Brassicae butterflies know through observation of other butterflies and wasps the probability that their eggs will be destroyed if they mate with an anti-aphrodisiac positive male, the the optimal strategy should the following:

$((1 - P(\text{wasp})) \cdot P(M_A))$ $P(M_a)$

If the probability of the anti-aphrodisiac male's offspring surviving is greater than the probability of the non-anti-aphrodisiac male's offspring surviving, then the female butterflies should mate with the anti-aphrodisiac butterflies.

Otherwise, the females should mate with the non-anti-aphrodisiac male butterflies.

Over time, this would affect the anti-aphrodisiac male populations and the non-anti-aphrodisiac male populations the most.