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The Problem:

Land a probe gently enough to minimize the distance from a predetermined location, move the probe to a new location whilst minimizing energy and how far the probe bounces, and also determining the smallest dimensions of an asteroid for possible landings.

Critical Information:

- Before the first contact with asteroid, there are no non-conservative forces acting on the probe. -Therefore we can say that the total energy is the sum of kinetic and potential energy.
- The necessary information of the asteroid such as mathematically estimated gravitational field $\boldsymbol{U}(\boldsymbol{r})$ (where $\boldsymbol{r}$ is the position vector from the center of mass of asteroid), asteroid's path, and high-resolution images(visible and thermal) are obtained through remote sensing data.

The Model:

1) **Landing Model**

For the problem of landing a probe on an asteroid, we assume as follow:

- The asteroid is rotating with respect to only one axis (assume z-axis) and the angular velocity **ω** is time-invariant about this axis.
- the probe will be released preferably from a point where the predetermined position can be seen clearly.

We consider the inertial frame of reference to be at the center of mass of the asteroid; all of the vector quantities in our calculations will be originated from this center of the mass. Our goal is to accommodate our releasing condition, the initial velocity $\boldsymbol{v}(\mathbf{0}) = \boldsymbol{v}_i$, with the rotating target so that the probe lands as close as possible to the target. Upon release with $\boldsymbol{r}(\mathbf{0}) = \boldsymbol{r}_i$, the position of the probe is governed by the equation:

$$\ddot{r} = \frac{1}{m}(-\nabla U(r) + F_{other}(r))$$

where F_{other} = External forces to consider such as radiation force from a light source, gas drag

m = the mass of the probe.

The solution, $\boldsymbol{r}$, of this equation will be a function of time and initial velocity, $\boldsymbol{v}_i$. On the other hand, the surface that is initially facing the rover will be rotating about the z-axis. From the image data, we can get the numerical positions of surface points or, hopefully, the estimated surface equation; $\boldsymbol{S}$. The new surface, S_{new} , that is rotated surface after releasing the probe will be given by:

$$S_{new} = R_z(\omega t) \times S_{initial} \quad \text{where} \quad R_z(\omega t) = \begin{bmatrix} cos(\omega t) & -sin(\omega t) & 0 \\ sin(\omega t) & cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} (rotation\ about\ z-axis)$$

$S_{initial}$ = initial surface estimation

In the above equation, the surface itself and the points on the surface are interchangeable; i.e., S_{new} with x_{new} and $S_{initial}$ with the predetermined target on the initial surface, $x_{initial}$. So, the equation we need to solve will be:

$$r(t,\ v_i) = x_{new} = R_z(\omega t) \times x_{old}$$

Our solution will be an implicit or explicit expression containing variables $\boldsymbol{v}_i$ and falling time t. From this solution alone, we have numerous choices of $\boldsymbol{v}_i$ and t that work for our scenario. However, we will narrow down the solution set by considering the next problem; landing as close as possible to the predesignated point while facing the bouncing problem after landing.

2) Bouncing process

First, we assume various scenarios for this process.

- The gravity of the asteroid is the prominent factor for ballistic motion on the asteroid's surface.
- The path of bouncing can be tracked as a straight line; i.e., total displacement is equal to the total distance traveled, and the angle of bouncing remains the same throughout the bouncing process.
- The surface of the asteroid reduces the same factor, k, of kinetic energy after each encounter with the probe.

Then, we have:

$$KE_i = k * KE_{i-1} \text{ where } k \in (0,\ 1)$$
$$\theta_i = \theta_{i-1}$$
$$R_i = \frac{v_i^2 \sin(2\theta_i)}{g} = \frac{k^{i-1}\ v_1^2 \sin(2\theta_1)}{g}$$
$$\text{After n bounces, } R_n = \sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \frac{k^{i-1}\ v_1^2 \sin(2\theta_1)}{g} = \frac{v_1^2 \sin(2\theta_1)}{g} * \frac{1-k^n}{1-k}$$
$$\text{After coming to rest, } R_{total} = \lim_{n \to \infty} R_n = \frac{v_1^2 \sin(2\theta_1)}{g(1\ -\ k)}$$

Thus, we can easily find the maximum range of one bouncing process can travel by letting $\theta_1 = \frac{\pi}{4}$and $v_1$slightly less than escape velocity.

In reality, R_{total} and $R_n$will be in a margin error after finite bounces. For example, when $R_{total} - R_n = 0.3 \text{ to } 0.5\ m$, then bounces after n bounces become insignificant. Then, the continuous function of n (although discrete) using $v_1$and θ_1 will be:

$$R_{total} - R_n = Error_{margin}$$

$$n = \frac{1}{\ln k} * \ln\left(1 - Error_{margin} \times \frac{g\ (1\ -\ k)}{v_1^2\ \sin\ (2\theta_1)}\right)$$

3) Landing as close as possible to the predesignated target

Considering our bouncing mechanism on the surface, we want the direction of the impact velocity to be perpendicular to the surface, S, and the reduced velocity after impact to be less than the escape velocity of the asteroid. Translating this to the mathematical expression:

$$\frac{\dot{r}(t_{fall})}{||\dot{r}(t_{fall})||} = -\nabla S$$

$$v_{escape} \geqq \sqrt{k}\,||\dot{r}(t_{fall})||$$

$$\begin{aligned} \text{where } \dot{r}(t_{fall}) &= \text{impact velocity} \\ S &= surface \\ v_{escape} &= \text{escape velocity} \end{aligned}$$

The left side of first equation is a function of t_{fall} and v_i because we haven't fully solved our position vector solution from the "Landing Model" section. However, we obtained an expression of $\boldsymbol{v}_i$ and t at the end of the mentioned section. Since we now have two equations with two unknowns, it would be possible to narrow down the choice of $\boldsymbol{v}_i$ such that the landing is as close as to the designated target.

4) Moving to a new position

We will use a spring to store the necessary amount of energy and release it so that the probe requires kinetic energy to move on the surface.

If the surface is locally flat between the start point and the endpoint, using one or more "bouncing process" is sufficient (A bouncing process may have many bounces).

5) Asteroid Dimensions

From part one we are able to find the impact velocity of a given probe based on any asteroid. So if we set the probes impact velocity equal to the asteroids escape velocity, which is dependent on mass and radius then we can say that minimizing the impact velocity allows for smaller dimensions.

$$V_{imp} = V_{esc}$$

$$V_{esc}^2 = \frac{2GM_{ast}}{R}$$

$$M_{Ast} = \rho * V$$

$$V = \frac{4}{3}\pi R^3$$

V -Volume

$$V_{imp}^2 = \frac{2G\rho\frac{4}{3}\pi R^3}{R}$$
$$= \frac{8}{3}G\rho\pi R^2$$

We can now conclude the smallest possible dimensions a probe can land on an asteroid would be

$$R = \sqrt{\frac{3V_{imp}^2}{8\pi G\rho}}$$

$$M = \rho\left(\frac{4}{3}\pi\left(\sqrt{\frac{3V_{imp}^2}{8\pi G\rho}}\right)^2\right)$$

when minimizing impact velocity.