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STUDENT VERSION
Earth's Climate–A Balancing Act

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STATEMENT

Earth's Climate System

How can we use mathematics to help us understand Earth's climate? Better yet, what do we even mean by climate, and how is it different than weather? Weather can be thought of as a mix of events that happen each day in our atmosphere . On the other hand, climate can be thought of as the long term atmospheric conditions in a region. That is, "climate is what we expect, weather is what we get." Earth's climate includes interactions among the atmosphere, hydrosphere (oceans, lakes and other bodies of water), geosphere (land surface), biosphere (all living things), and the cryosphere (snow and ice). The climate system is the exchange of energies and moisture between these spheres (see Figure **??**).

A climate model is a description of this system in mathematical terms. Climate models are used in computational simulations to explore the behavior of the system under various forcing scenarios. In particular, these models can be used to determine causes of Earth's climate change. Climate can be best described as the statistics of weather. Accordingly, climate change refers to changes in the statistics of weather over time.

In this activity, we introduce a way to approach Earth's "climate system" mathematically through a global *energy balance model* (EBM). Energy balance models are climate models that try to predict the average surface temperature of the Earth from solar radiation, emission of radiation to outer space, and Earth's energy absorption and greenhouse effects. A global EBM summarizes the state of the Earth's climate system in a single variable–the temperature at the Earth's surface averaged over the entire globe. We begin with the simplest model and assume that Earth is a homogeneous solid sphere. That is, we ignore differences in topography (altitude), differences in the atmosphere's composition (such as clouds), differences among continents and oceans, and so on. Because we are not considering any spatial variations, these models are sometimes referred to as *zero-dimensional energy balance models.*

Observation

The climate system is powered by the sun, which emits radiation in the ultraviolent (UV) regime (wavelength less than 0.4 μm). This energy reaches the Earth's surface, where it is converted by physical, chemical, and biological processes to radiation in the infrared (IR) regime (wavelength greater than 5 μm). This IR radiation is then re-emitted into space. If the Earth's climate is in *equilibrium*, the average temperature of Earth's surface does not change, so the amount of energy received must equal the amount of energy re-emitted.

To build the model, we need to introduce the following units, variables and physical parameters.

Units

- Length: meter (m), a μm is a micrometer = 0.001 mm.
- Energy: watt (W). 1 watt = 1 joule per second = $1 \frac{\text{kg}\cdot\text{m}^2}{\text{sec}^3}$. A joule is the unit of energy used by the International Standard of Units (SI). It is defined as the amount of work done on a body over a distance of one meter.
- Temperature: kelvin (K). An object whose temperature is 0 K has no thermal energy, i.e. 0 K is *absolute zero.* The Kelvin scale is closely related to the Celsius scale. The magnitude of a degree in the Celsius scale is the same as the magnitude of a kelvin in the Kelvin scale, but the zero point is different. Water freezes at the zero point in Celsius and at 273.15 K. Thus, the Kelvin scale is the Celsius scale plus 273.15.

Variable

- T, the temperature of the Earth's surface average over the entire globe.

Physical parameters

- R, the radius of the Earth.
- S, the energy flux density–the rate of transfer of energy through a surface or rate of energy transfer per unit area. Through satellite observations, $S = 1367.6$ Wm^{-2}.
- σ (sigma), Stefan-Boltzmann constant; its value is given by $\sigma = 5.67 \cdot 10^{-8}\text{Wm}^{-2}\text{K}^{-4}$. The Stefan-Boltzmann constant is the constant of proportionality in the Stefan-Boltzmann law: "the total energy radiated per unit surface area of a black body across all wavelengths per

unit time is directly proportional to the fourth power of the black body's thermodynamic temperature T". By *black body* we mean a physical body that absorbs all incident electromagnetic radiation.

Next, we make use of the following assumptions:

Assumptions

- Viewed from the sun, the Earth is a disk.
- The area of the disk as seen by the sun is πR^2.
- Recalling that the energy flux density is S, the amount of energy flowing through the disk (i.e. reaching the Earth) is

$$\text{Incoming energy (W)} : E_{in} = \pi R^2 S. \tag{1}$$

- All bodies radiate energy in the form of electromagnetic radiation.
- In physics, it is shown that for "black-body radiation" the temperature dependence is given by the Stefan–Boltzmann law (see above, in units of Wm^{-2}),

$$F_{SB}(T) = \sigma T^4. \tag{2}$$

- The area of the Earth's surface is $4\pi R^2$.
- The amount of energy radiated out by the Earth is

$$\text{Outgoing energy (W)} : E_{out} = 4\pi R^2 \sigma T^4. \tag{3}$$

Problem 1

Recalling that energy is measured in watts, verify that the units of E_{in} and E_{out} are in watts (W). Note that if the incoming energy is greater than the outgoing energy, the Earth's temperature will increase. Likewise, if the outgoing energy is greater than the incoming energy, the Earth's temperature will decrease. We are interested in the case where the Earth's temperature remains constant. That is, Earth is in *thermal equilibrium*. Determine the temperature T for which Earth is in thermal equilibrium, i.e. $E_{in} = E_{out}$ (energy balance equation, see (1) and (3)). The known average temperature of the Earth is about 16 degrees Celsius (or 287.7K). How well does your answer compare with the known average temperature?

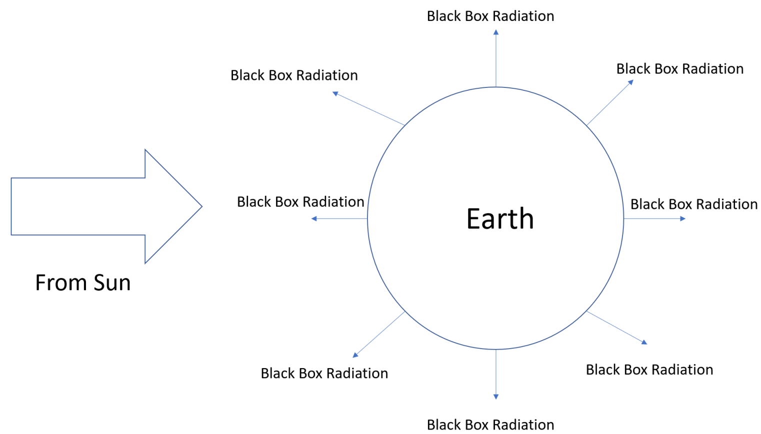

Figure 1. A Conceptual model of Earth's climate system: incoming sunlight and outgoing heat.

Problem 2

You probably noticed that your answer in Problem 1 was pretty far off from 16 degrees Celsius. It turns out that the model posed in Problem 1 was too simple and omitted too many different factors. For instance, we did not consider the fact that snow, ice, and clouds can reflect a significant amount of incoming energy from the Sun. This fraction of energy that is reflected back into space before it reaches the Earth's surface is called the (planetary) *albedo*. Let α denote the albedo, so that the remaining fraction $1 - \alpha$ (sometimes called the *co-albedo*) of the incoming solar radiation will reach the Earth's surface. Adding the effects of albedo to E_{in}, adjust the energy balance equation and determine the temperature T for which Earth is in thermal equilibrium. A typical value for Earth's *average* albedo is $\alpha = 0.30$. This means that about 70% of the incoming energy is absorbed by the Earth's surface. How well does your answer compare with the known average temperature? Would you say that this new model is an improvement over the previous model? Why/Why not?

Problem 3

You might have noticed that your answer in Problem 2 is even worse than the solution in Problem 1–even though this new model is more physically relevant! Reminding ourselves that modeling is an iterative process, rather than throw away the albedo introduced in Problem 2, we add another factor which has a significant effect on the global equilibrium temperature. A good portion of the difference between your answer in Problem 2 and the average global temperature can be attributed to the *greenhouse effect* of Earth's atmosphere. That is, we include the effects of greenhouse gases (like carbon dioxide and methane), water, dust, and aerosols on the atmosphere. The chemical properties of these greenhouse gases have a significant effect on the atmosphere by reducing the Stefan-Boltzmann law by some factor. This, in turn, affects the outgoing energy.

Let ϵ denote the *greenhouse factor* which is an artificial parameter used to model the effect of greenhouse gases on the permittivity of the atmosphere. While the value of ϵ is unknown, we will assume that $0 < \epsilon < 1$.

(a) Write a new energy balance equation which incorporates **albedo** and the **greenhouse effect**.

(b) Because the value of ϵ is unknown, it is not possible to determine the equilibrium temperature T. However, we can work backwards to determine the value ϵ^* so that the equilibrium temperature is $T = 287.7K$. Determine this value ϵ^*.

(c) Suppose that the combined effects of greenhouse gases, dust, and aerosols reduces the parameter ϵ so that $\epsilon < \epsilon^*$. What happens to the equilibrium temperature? Is this what you expected to have happen? *Remark: It may be helpful to read up on the Greenhouse Effect. See [3] for instance.*

Problem 4–A Differential Equation is Born!

Earlier we mentioned that if the incoming energy is greater than the outgoing energy, the Earth's temperature will increase. Likewise, if the outgoing energy is greater than the incoming energy, the Earth's temperature will decrease. Suppose the temperature is increasing. Will the temperature continue to increase, or will the temperature eventually level off? How fast will the temperature change? To answer these types of questions, we must adjust our model so that it allows the temperature to change over time. Perhaps the simplest model is one that assumes that the temperature changes at a rate proportional to the energy imbalance. Rewrite the last sentence as a mathematical equation using E_{in} and E_{out}.

Problem 5

It is traditional to formulate the differential equation of temperature evolution in terms of energy densities (Wm^{-2}). Right now, the right hand side of your differential equation is given in terms of E_{in} and E_{out} which are energies measured in watts (W). To convert these values to energy densities, we can divide by the Earth's surface area (πR^2). In terms of energy densities, the *temperature evolution equation* becomes

$$C\frac{dT}{dt} = \frac{1}{4}(1-\alpha)S - \epsilon\sigma T^4, \tag{4}$$

where C is the *planetary heat capacity* which connects the rate of change of the temperature to energy densities and is the amount of energy needed to raise the temperature of the planet by 1 K. Note that even in this new scenario, if $E_{in} = E_{out}$, the derivative is zero indicating that Earth is in thermal equilibrium. With this differential equation (and taking $\alpha = .3, \epsilon = 0.66, C = 1$):

(a) Determine the effect of $E_{in} > E_{out}$ on the global average temperature T. Does your answer make sense physically?

(b) Determine the effect of $E_{in} < E_{out}$ on the global average temperature T. Does your answer make sense physically?

(c) Suppose that the current temperature is 350 K. Do you expect the temperature to increase, decrease, or remain the same?

(d) Suppose that the current temperature is 250 K. Do you expect the temperature to increase, decrease, or remain the same?

(e) What does this suggest about the *stability* of the equilibrium point? Defend your answer by performing a phase line analysis. Assume that a reasonable domain for T is $[200, 400]$.

(f) Now suppose that $\epsilon = 0.5$. Determine the stability of the equilibrium point by performing a phase line analysis. How does this analysis compare to the previous problem?

Problem 6

So far, we have assumed that the albedo is constant and independent of the surface temperature. However, this assumption does not account for the fact that when the surface temperature is sufficiently low, water turns to ice and increases the ability for Earth to reflect incoming energy from the Sun. Thus, we should consider a temperature-dependent albedo with the following constraint:

$$\alpha(T) \approx \begin{cases} 0.7 & T < 250, \\ 0.3 & T > 280. \end{cases} \tag{5}$$

This allows us to incorporate the assumption that when T is low enough, water turns to ice and increases the albedo. We are now in a position to adjust our temperature evolution equation (4) by replacing α with a monotonically decreasing function $\alpha(T)$ that connects the value 0.7 at $T \approx 250K$ with the value 0.3 if $T > 280K$. There are several ways of accomplishing this. One such way is show in Figure 2.

$$\alpha(T) = 0.5 - 0.2 \cdot \tanh\left(\frac{T - 265}{10}\right), \tag{6}$$

where the *hyperbolic tangent function*, $\tanh(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}$.

Our temperature evolution equation now becomes

$$C\frac{dT}{dt} = \frac{1}{4}(1 - \alpha(T))S - \epsilon\sigma T^4, \tag{7}$$

with $\alpha(T)$ given by (6). We would like to perform a similar phase line analysis as in the previous problem. Since this is a new ODE, we begin by finding the equilibrium points. Noticing that the right-hand side of the differential equation is a complicated nonlinear function (of T–all other values are known), we must estimate the equilibrium points using a root finding method. The equilibrium points for (7) occur when $E_{in} = E_{out}$ (Why?). Since E_{in} and E_{out} are both functions of time, we

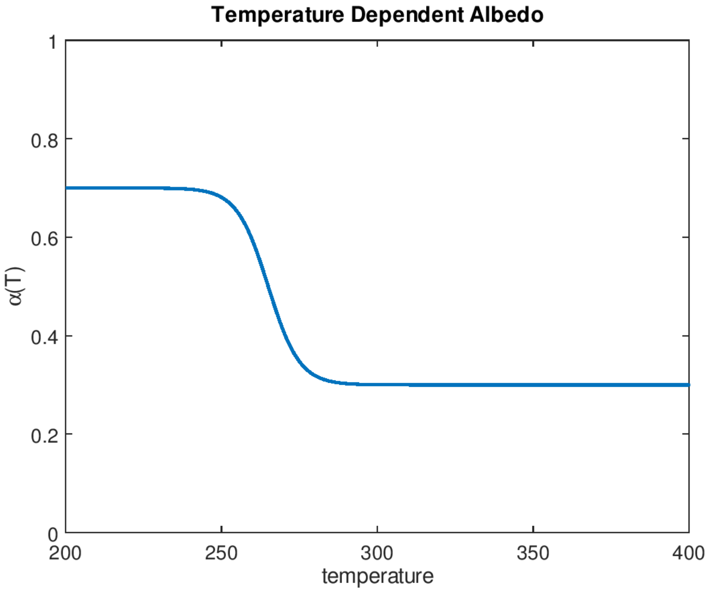

Figure 2. Graph of (6).

can plot E_{in} and E_{out} to see that (7) has three equilibrium points. (See Figure 3.) We will use the "Solver" add-in in Excel to find the values of T so that $f(T) = \frac{1}{4}(1 - \alpha(T))S - \epsilon\sigma T^4 = 0$. Before we access Solver, we must first input $f(T)$ into Excel. See Figure 4.

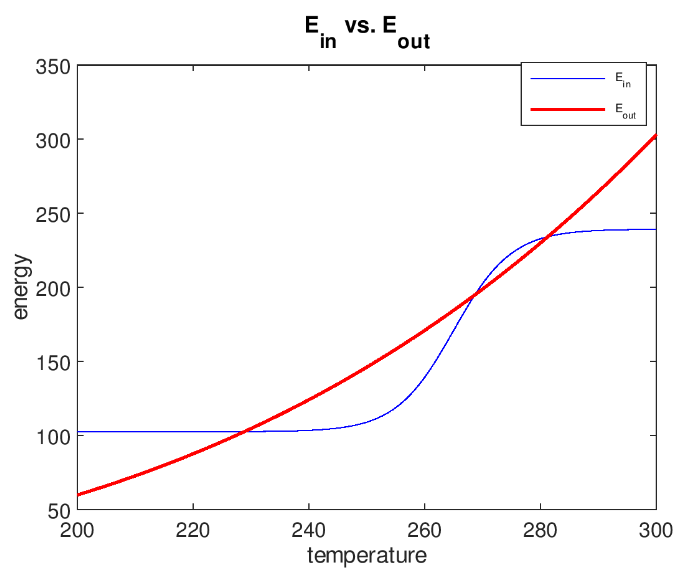

Figure 3. E_{in} vs. E_{out}.

In Figure 4, A1 $= f(200)$.

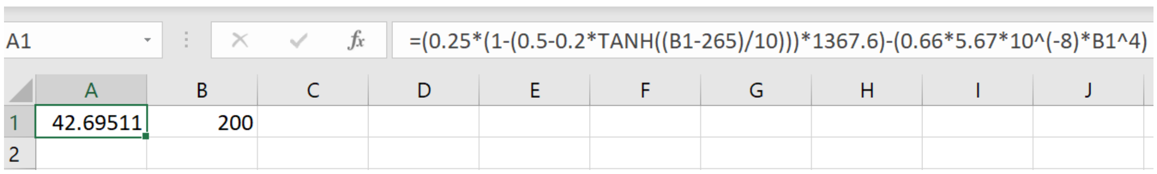

Figure 4. Inputting $f(T)$ into Excel

Using Excel's Solver

The procedure for using Excel's Solver is as follows:

1. You can access Solver in one of two ways, depending on which version of Excel is being used. Under the "Tools" menu select "Solver". A new pop-up window will appear. *Remark*: If you do not see this as an option, the add-in will need to be installed. To access Solver, select "Add-Ins" under the "Tools" menu and check the solver add-in.

   Otherwise, you can access Solver in the Analysis group under the "Data" tab. *Remark*: If you do not see this as an option, the add-in will need to be installed. To access Solver, go to File > Options. Click Add-Ins, and then in the Manage box, select Excel Add-ins. Click go. In the Add-Ins available box, select the Solver Add-in check box, and then click OK. After you load the Solver Add-in, the Solver command is available in the Analysis group on the Data tab. See Figure 5.

2. In the box labeled "Set Objective:" enter the reference of the cell into which you typed the formula. In our example, we would type A1.

3. Click the "Value of:" button. Enter your target value in the "Value Of:" box. In our example, we would type 0. (This is the right hand side of the nonlinear equation that we are trying to solve.)

4. In the "By Changing Variable Cells:" box type the formula's reference cell. In our example, we would type B1.

5. Click "Solve." Excel will change both cells accordingly.

Remark: To identify all three equilibrium points, you must select a value for A1 that is fairly close to the equilibrium point you wish to find. Use Figure 3 to identify a good guess for A1 for each equilibrium point. Once you have determined the three equilibrium points, perform a phase line analysis for each equilibrium point and classify each point as a source, sink, or node. Explain the physical relevancy of your results being sure to describe the kind of climate each equilibrium point dictates. What would it have been like to live on Earth in each case?

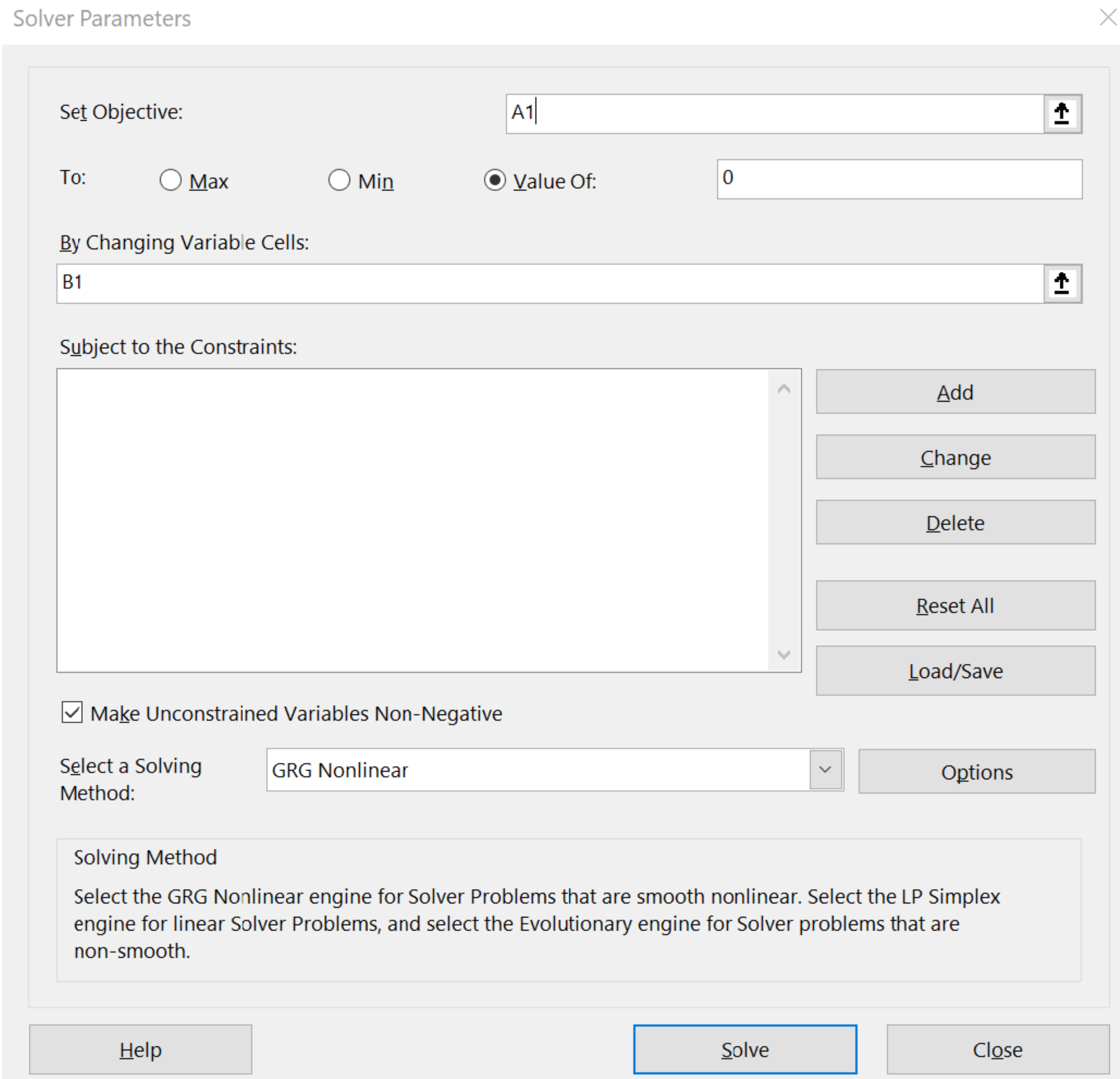

Figure 5. Excel's Solver Tool.

Problem 7

Finally, we consider a possible improvement of our model by incorporating data collected by satellites about the energy radiated out by the Earth. While we have been assuming that Earth radiates like a black body (so that the outgoing radiation follows the Stefan-Boltzmann law), Mikhail Budyko and William Sellers [2] proposed a different expression for the outgoing radiation. They proposed the following model for outgoing energy:

$$E_{out}(T) = A + BT, \tag{8}$$

where A and B are constants. North and Coakley [1] were able to validate this model using this observational data; in particular, it was estimated that $A = 203.3$ Wm^{-2} and $B = 2.09$ Wm^{-2}deg^{-1}. Here, temperatures are measured in Celsius. Thus, when we incorporate this result into our temperature evolution equation, we use $T - 273.15$ instead of just T. (Why?)

(a) Repeat Problem 6 with the new temperature evolution model:

$$C\frac{dT}{dt} = \frac{1}{4}(1 - \alpha(T))S - (A + B(T - 273.15)), \tag{9}$$

where $\alpha(T)$ is given in (6) and A and B are given above. Compare your results with those obtained in Problem 6.

(b) One way of directly comparing (7) with (9) is by computing a linear expansion of $E^*_{out}(T) = \sigma(273.15 + T)^4$ about $T = 0$ (accounting for the fact that A and B were obtained with temperatures measured in Celsius). Compute the linearization of $E^*_{out}(T)$ and determine constants A^* and B^* so that $E^*_{out}(T) \approx A^* + B^*T$. Compare your results with (8).

Wrap Up

Starting with the simple observation that the global average temperature at the Earth's surface increases if the amount of energy reaching the Earth exceeds the amount of energy emitted by the Earth and released into the stratosphere (and vice versa), you were able to develop models which predict the global mean temperature of Earth. More than that, you observed three different equilibrium states for the global mean temperature. How does this relate to Earth's current climate? One of the states corresponds to the current climate, while another equilibrium state was found to be unstable. The third stable equilibrium state corresponds to a deep-freeze climate, where the Earth would have been completely covered with snow and ice. In fact, this equilibrium state corresponds to a complete glaciation of the Earth, with all oceans frozen to a depth of several kilometers and almost the entire planet is covered in ice. This dramatically different Earth, for which no life could have existed, is sometimes referred to as *Snowball Earth*. There is some debate about whether Earth was completely covered with snow and ice or if there were still some "slushy" spots that could have allowed for some organisms to survive.

REFERENCES

[1] North, G. R. and J. A. Coakley. 1979. Differences between seasonal and mean annual energy balance model calculations of climate and climate sensitivity. *Journal of the Atmospheric Sciences*. 36: 1189-1204.

[2] Budyko, M. I. 1969. The effect of solar radiation variations on the climate of the Earth. *Tellus*. 21: 611-619.

[3] "The Greenhouse Effect". University Corporation for Atmospheric Research. `https://scied.ucar.edu/longcontent/greenhouse-effect`. Accessed 21 July 2019.