M&M Death and Immigration

(Modification of the STUDENT VERSION 1-001-S-M&M Death and Immigration by Brian Winkel, Director SIMIODE, Cornwall NY USA)

This is a modification of the student version of SIMIODE scenario *M&M Death and Immigration*. A Teacher Version as well as the original Student Version of the scenario are available at SIMIODE.org. The original published version of the scenario includes suggestions for the use of both *Mathematica* and an *Excel* spreadsheet. This adaptation is designed to fit in the time constraints of this workshop. It is an abbreviated version of the original scenario and is also influenced by the modified versions of the scenario written by Dina Yagodich and Karen Bliss

**STATEMENT**

We will use M&M candies to conduct two simulations. The first simulation will be used to investigate a death model without immigration and the second simulation will investigate a death model with immigration. In each simulation, M&M candies will be tossed onto a plate. Those candies that land with the letter “m” facing upward will die. The remaining M&M candies will continue to the next generation. In the second simulation (with immigration) a fixed number of M&M candies will be added back in after each death phase. Each simulation will include brief data visualization and model building components. Suggestions for customizing further analysis of the results are included.

**I. SIMULATION 1: DEATH WITHOUT IMMIGRATION**

**A. Set-Up**

1. Break into groups. Each group will be provided with at least 50 M&M candies, a paper plate and two paper cups, one of which is marked with the letter “X.”
2. Place 50 candies into the plain cup and the remaining candies in the cup marked with an X.
3. Read the simulation directions below but DO NOT begin the simulation just yet
4. Describe what you expect to happen. In particular, address:
   a) How many live M&M candies will there be at the end of the experiment?
   b) How many generations (iterations) do you think it will take to reach this number?
B. Simulation

1. Toss the M&Ms gently onto the plate.
2. Remove the M&Ms with the letter “m” facing up (they die) and place them in the cup marked with an X. Look carefully—the yellow ones can be hard to read.
3. Count the number of M&Ms remaining from that generation and record the data in Table 1 (shown to the right).
4. Repeat until you are satisfied that you have reached the final number or end of the experiment.

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5. Compare your description (from Part A) of what you thought would happen with what actually happened.

C. Building a Mathematical Model

1. On the axes provided below, plot the simulation data that you recorded in Table 1.
2. Next, let \( a(n) \) represent the number of M&Ms alive at generation (iteration) \( n \). What values does \( n \) take? What is \( a(0) \)? What should be the value of \( a(1) \) in your model? What assumptions or generalizations are you making to determine \( a(1) \)?

3. Based on the observations and your assumptions, produce a reasonable formula for \( a(n) \) in terms of \( n \) for \( n = 0, 1, 2, \ldots \).

4. How plausible is your discrete function model for \( a(n) \)? Are the assumptions or generalizations you used reasonable? Why? What is the long-term behavior of your model (i.e. what is \( a(n) \) as \( n \to \infty \))?  

5. Compare your model to your data. On your plot of the data, also plot the M&M population predicted by your model at each generation (i.e. plot \( a(0), a(1), a(2), \ldots \)). How well does your model predict the results from your simulation?

6. Compare your model to the models developed by other groups.

7. How can you measure your “success” as a modeler in this situation?
II. SIMULATION 2: DEATH WITH IMMIGRATION

A. Set-Up
1. Choose a number between 1 and 50 for your group. This will be the number of M&M candies that your group will begin with in the simulation.
2. Place your beginning number of candies into the plain cup and the remaining candies in the cup marked with an X.
3. Read the simulation directions below but DO NOT begin the simulation just yet.
4. Describe what you expect to happen. In particular, address:
   a) How many live M&M candies will there be at the end of the experiment?
   b) How many generations do you think it will take to reach this number?
   c) Other groups may begin with different numbers of M&Ms. Make conjectures about their outcomes as well. What do you think will happen if their starting number of M&Ms is 0? 25? Or 50?

B. Simulation
1. Toss the M&Ms gently onto the plate.
2. Remove the M&Ms with the letter “m” facing up (they die) and place them in the cup marked with an X.
3. Add 10 new “immigrant” M&Ms. You can use the M&Ms from the cup marked with an X.
4. Count the number of M&Ms remaining from that generation including the immigrant M&Ms and record the data in Table 2 (shown to the right).
5. Repeat until you are satisfied that you have reached a final number or end of the experiment.

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6. Compare your description (from Part A) of what you thought would happen with what actually happened.
C. Building a Mathematical Model

1. On the axes provided below, plot the simulation data that you recorded in Table 2.

2. Next, let \( b(n) \) represent the number of M&Ms alive at generation (iteration) \( n \). What is \( b(0) \)? What should be the value of \( b(1) \) in your model? What assumptions or generalizations are you making to determine \( b(1) \)?

3. What should be the value of \( b(2) \) in your model? \( b(3) \)? Try to create a reasonable formula for \( b(n) \) in terms of \( n \) for \( n = 0, 1, 2, \ldots \).

4. It can be difficult to come up with a formula directly in terms of \( n \) when developing a model. Instead, we can focus on how to get from one generation to the next.
   a. What is the M&M population in generation “\( n + 1 \)” in terms of the M&M population in the \( n^{th} \) generation? Write a formula for \( b(n + 1) \) in terms of \( b(n) \). Be sure you have accounted for immigration!
   b. Now, write and simplify the difference \( b(n + 1) - b(n) = \Delta b(n) \) to model the population change at each generation.

5. How do you expect \( \Delta b(n) \) to change as \( n \) grows large? Explain your reasoning. Now, compare your model for population change at each generation to the actual change between generations in your data. Does your choice of initial population affect the long-term behavior of \( \Delta b(n) \)?
III. Ideas for Further Analysis

There are many options for continuing this classroom activity depending on the content you wish to emphasize. Some possibilities are listed below.

Discuss how you might continue this scenario to introduce or address additional course topics such as those listed. Give specific details on possible implementation.

1. Consider variations to the simulation.
   a. Change the constant number of immigrants added at each generation and explore how that affects the long-term behavior. What if the number of immigrants added at each generation is not constant but changes in a predictable way?
   b. Modify the criteria used for determining death in the simulation.

2. Compare the data.
   a. Explore all the data collected in class by using Excel to plot the data from each group.
   b. Discuss the effects of the different initial populations on the final population of M&Ms.
   c. Relate the solution curves and initial conditions to the direction field (see idea 3 below).

3. Investigate the direction field.
   a. Develop the idea for a direction field by considering \( \Delta b / \Delta n \).
   b. Explore the effect of different initial conditions (beginning values of M&Ms).
   c. Examine the long-term behavior and discuss stable and unstable equilibrium solutions.

4. Develop the corresponding differential equation model.
   a. Discuss the relationship between the difference equation and the corresponding differential equation \( \left( \frac{\Delta b}{\Delta n} \approx \frac{db}{dt} \right) \). This topic is included in the original SIMIODE scenario.
   b. Write the differential equation model.
   c. Discuss techniques for solving the linear first-order non-homogeneous differential equation.

5. Construct a numerical solution.
   a. Use Euler’s Method or another numerical technique to approximate a solution.
   b. Compare a plot of the numerical solution to the plot of the data.

6. Determine a closed form discrete solution.
   a. Revisit Part C. Building a Mathematical Model for the Death and Immigration situation and determine a closed form for \( b(n) \) (i.e. write \( b \) in terms of \( n \) only).
   b. Compare this closed form solution to the solution for the differential equation.

- Discuss how you might continue this scenario to introduce or address additional course topics such as those listed. Give specific details on possible implementation.

- Can you think of other topics from an Ordinary Differential Equations course that could be introduced or explored using this scenario?